Mathematics 3790H – Analysis I: Introduction to analysis
Trent University, Fall 2010

Solutions to Assignment #4

The integral form of the remainder of a Taylor series

In what follows, let us suppose that c is a real number and $f(x)$ is a function such that $f^{(n)}(x)$ is defined and continuous for all $n \geq 0$ and for all x in some open interval I containing c. Recall that for $n \geq 0$, the Taylor polynomial of degree n of $f(x)$ at c is

$$\begin{aligned} T_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \,, \end{aligned}$$

and that the corresponding remainder term is $R_n(x) = f(x) - T_n(x)$. In what follows, we will assume that every x we encounter is in the interval I.

1. Use the Fundamental Theorem of Calculus to show that

$$R_0(x) = \int_c^x f'(t)\, dt \,. \quad [1]$$

Solution. Since $T_0(x) = f(c)$, $\displaystyle\int_c^x f'(t)\, dt = f(x) - f(c) = f(x) - T_0(x) = R_0(x)$. ■

Note: It follows from the definitions of $T_n(x)$ and $R_n(x)$ that

$$\begin{aligned} R_{n+1}(x) &= f(x) - T_{n+1}(x) \\ &= f(x) - \left[T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} \right] \\ &= [f(x) - T_n(x)] - \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} \\ &= R_n(x) - \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} \end{aligned}$$

for all $n \geq 0$. We will use this observation below.

2. Use the formula in **1** and integration by parts to show that

$$R_1(x) = \int_c^x f''(t)(x-t)\, dt \,. \quad [2]$$

Hint: Use the parts $u = f'(t)$ and $v = t - x$ ...

SOLUTION. We start with the integral in **1** and apply parts with $u = f'(t)$ and $v = t - x$, so $du = f''(t)$ and $dv = dt$.

$$\begin{aligned}\int_a^x f'(t)\,dt &= f'(t)(t-x)\big|_c^x - \int_c^x f''(t)(t-x)\,dt \\ &= f'(x)(x-x) - f'(c)(c-x) + \int_c^x f''(t)(x-t)\,dt \\ &= f'(c)(x-c) + \int_c^x f''(t)(x-t)\,dt\end{aligned}$$

It follows that

$$\begin{aligned}\int_c^x f''(t)(x-t)\,dt &= \int_c^x f'(t)\,dt - f'(c)(x-c) \\ &= R_0(x) - f'(c)(x-c) \qquad \text{(By } \mathbf{1}.) \\ &= R_1(x) \qquad \text{(By the NOTE above.)}\,,\end{aligned}$$

as desired. ■

3. Use the formula in **2** and integration by parts to show that

$$R_2(x) = \frac{1}{2}\int_c^x f^{(3)}(t)(x-t)^2\,dt\,. \quad [2]$$

SOLUTION. We start with the integral in **2** and apply parts with $u = f''(t)$ and $v = \frac{1}{2}(t-x)^2$, so $du = f^{(3)}(t)$ and $dv = (t-x)dt$.

$$\begin{aligned}\int_c^x f''(t)(x-t)\,dt &= -\int_c^x f''(t)(t-x)\,dt \\ &= -\left(\frac{f''(t)}{2}(t-x)^2\bigg|_c^x - \int_c^x \frac{f^{(3)}(t)}{2}(t-x)^2\,dt\right) \\ &= -\left(\frac{f''(x)}{2}(x-x)^2 - \frac{f''(c)}{2}(c-x)^2 - \int_c^x \frac{f^{(3)}(t)}{2}(t-x)^2\,dt\right) \\ &= \frac{f''(c)}{2}(x-c)^2 + \int_c^x \frac{f^{(3)}(t)}{2}(x-t)^2\,dt\end{aligned}$$

It follows that

$$\begin{aligned}\int_c^x \frac{f^{(3)}(t)}{2}(x-t)^2\,dt &= \int_c^x f''(t)(x-t)\,dt - \frac{f''(c)}{2}(x-c)^2 \\ &= R_1(x) - \frac{f''(c)}{2}(x-c)^2 \qquad \text{(By } \mathbf{2}.) \\ &= R_2(x) \qquad \text{(By the NOTE above.)}\,,\end{aligned}$$

as desired. ■

4. Use induction to show that

$$R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\,dt\,. \quad [5]$$

SOLUTION. As instructed, we will proceed by induction on n.

Base Step. ($n = 0$) We need to check that $R_0(x) = \displaystyle\int_c^x \frac{f^{(0+1)}(t)}{0!}(x-t)^0\,dt = \int_c^x f'(t)\,dt$. This is just problem **1**.

Induction Hypothesis. ($n = k$) Assume that$R_k(x) = \displaystyle\int_c^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt$.

Induction Step. ($n = k+1$) By the Induction Hypothesis and the NOTE above,

$$R_{k+1}(x) = \int_c^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt - \frac{f^{(k+1)}(c)}{(k+1)!}(x-c)^{k+1}$$

We will now apply integration by parts to the integral $\displaystyle\int_c^x \frac{f^{(k+2)}(t)}{(k+1)!}(x-t)^{k+1}\,dt$. Bearing in mind that t is the variable of integration, the parts are:

$$\begin{array}{rlrl} u &= (x-t)^{k+1} & dv &= \frac{f^{(k+2)}(t)}{(k+1)!}\,dt \\ du &= -(k+1)(x-t)^k\,dt & v &= \frac{f^{(k+1)}(t)}{(k+1)!} \end{array}$$

Here goes!

$$\begin{aligned}
&\int_c^x \frac{f^{(k+2)}(t)}{(k+1)!}(x-t)^{k+1}\,dt = \int_c^x u\,dv = uv\big|_a^x - \int_c^x v\,du \\
&= (x-t)^{k+1}\cdot\frac{f^{(k+1)}(t)}{(k+1)!}\bigg|_c^x - \int_c^x \frac{f^{(k+1)}(t)}{(k+1)!}\cdot(-1)(k+1)(x-t)^k\,dt \\
&= \left[0 - \frac{f^{(k+1)}(c)}{(k+1)!}(x-c)^{k+1}\right] - (-1)\int_c^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt \\
&= \int_c^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt - \frac{f^{(k+1)}(c)}{(k+1)!}(x-c)^{k+1}
\end{aligned}$$

Putting the two previous paragraphs together, we get

$$\begin{aligned}
R_{k+1}(x) &= \int_c^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt - \frac{f^{(k+1)}(c)}{(k+1)!}(x-c)^{k+1} \\
&= \int_c^x \frac{f^{(k+2)}(t)}{(k+1)!}(x-t)^{k+1}\,dt\,,
\end{aligned}$$

as desired.

It follows by induction that $R_n(x) = \displaystyle\int_c^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$ for $n \geq 0$. ■