Mathematics 3790H – Analysis I: Introduction to analysis
Trent University, Fall 2010

Solutions to Assignment #3

A slice of π

1. Verify that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{16n^2+16n+3}$ converges absolutely. *[4]*

Solution. Observe that the terms $\frac{1}{16n^2+16n+3}$ are all positive. Since $16n^2+16n+3 \geq 16n^2 \geq n^2$ for $n \geq 1$, we have

$$\frac{1}{16n^2+16n+3} \leq \frac{1}{16n^2} \leq \frac{1}{n^2}$$

for $n \geq 1$ as well. Hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{16n^2+16n+3}$ converges by comparison with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. (This series, in turn, converges by the p-test, among other possibilities.) Thus $\displaystyle\sum_{n=0}^{\infty} \frac{1}{16n^2+16n+3} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{16n^2+16n+3}$ must converge, too. ■

2. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{16n^2+16n+3} = \frac{\pi}{4}$. *[6]*

Hint: Start with the Taylor series at 0 of $\arctan(x)$ and use the fact that $\arctan(1) = \frac{\pi}{4}$. You'll need to do a little algebra, too.

Solution. Recall from class – or wherever! – that the Taylor series about 0 of $\arctan(x)$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots,$$

and converges for $x \in (-1, 1]$. Since $\tan(\pi/4) = 1$, it follows that:

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

We will consolidate each pair of successive terms of this series:

$$\begin{aligned}
\frac{\pi}{4} &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11}\right) + \cdots \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2(2k)+1} - \frac{1}{2(2k+1)+1}\right) \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+3}\right) \\
&= \sum_{k=0}^{\infty} \frac{(4k+3)-(4k+1)}{(4k+1)(4k+3)} \\
&= \sum_{k=0}^{\infty} \frac{2}{16k^2+16k+3} \\
&= 2\sum_{k=0}^{\infty} \frac{1}{16k^2+16k+3}
\end{aligned}$$

It follows that $\sum_{n=0}^{\infty} \frac{1}{16n^2+16n+3} = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}$. [Oops! My bad!] ■