Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2010

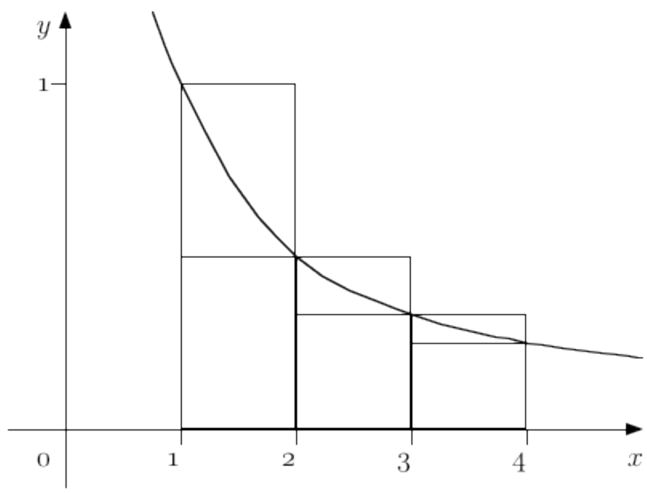
Solutions to Assignment #1

Euler's Constant

Euler's constant* is the real number γ defined by:

$$\gamma = \lim_{n\to\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k}\right) - \ln(n)\right] = \lim_{n\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln(n)\right]$$

Since $\ln(n) = \int_1^n \frac{1}{x}\,dx$, we can think of γ as a sum of areas: for each $k \geq 1$, consider the area of the rectangle of height $\frac{1}{k}$ with base the interval $[k, k+1]$ with the part below the curve $y = \frac{1}{x}$ taken away.



Your task is to show that the definition of Euler's constant makes sense.

1. Show that $\displaystyle\lim_{n\to\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k}\right) - \ln(n)\right]$ exists. *[7]*

Hint: For each rectangle of height $\frac{1}{k}$ with base the interval $[k, k+1]$ take away the part that lies below $y = \frac{1}{k+1}$.

SOLUTION. This is a little easier if we redefine how we get γ just a little bit.

Claim: $\displaystyle\lim_{n\to\infty}\left[\left(\sum_{k=1}^{n-1}\frac{1}{k}\right) - \ln(n)\right] = \lim_{n\to\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k}\right) - \ln(n)\right]$

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* It is traditionally denoted by γ and sometimes called the Euler-Mascheroni constant. In case you're curious, $\gamma = 0.5772156649\ldots$ It is unknown whether γ is rational or not.

Proof: Note that

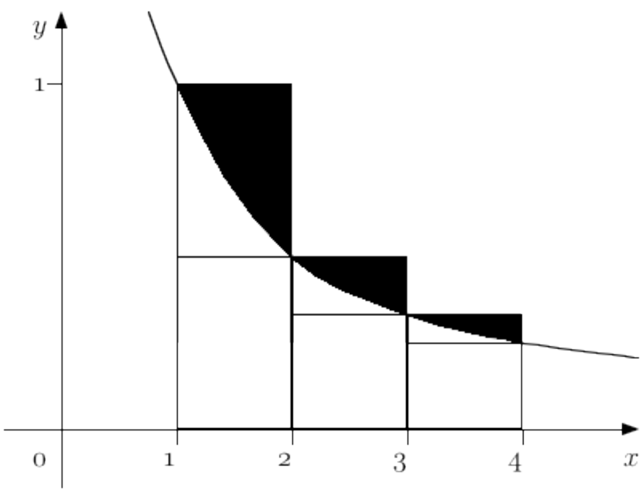
$$\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln(n)\right]-\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}-\ln(n)\right]=\frac{1}{n},$$

and $\frac{1}{n}\to 0$ as $n\to\infty$, so $\lim_{n\to\infty}\left[\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)-\ln(n)\right]=\lim_{n\to\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k}\right)-\ln(n)\right]$, so long as either limit exists. (This is much like the Cauchy condition for the convergence of sequences.) $\square$

With the claim in hand, we can really think of γ as a sum of areas:

$$\begin{aligned}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)-\ln(n)&=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}-\ln(n)\\&=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}-\int_1^n\frac{1}{x}\,dx\\&=\left(1-\int_1^2\frac{1}{x}\,dx\right)+\left(\frac{1}{2}-\int_2^3\frac{1}{x}\,dx\right)+\cdots+\left(\frac{1}{n-1}-\int_{n-1}^n\frac{1}{x}\,dx\right)\end{aligned}$$

See the picture below of the first three of the areas in question.



It should be clear that each of the areas $\frac{1}{k}-\int_k^{k+1}\frac{1}{x}\,dx$ is positive, and so, by the Monotone Convergence Theorem, all we need to do is show that their sum is bounded above to ensure that

$$\lim_{n\to\infty}\left[\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)-\ln(n)\right]=\lim_{n\to\infty}\left[\sum_{k=1}^{n-1}\left(\frac{1}{k}-\int_k^{k+1}\frac{1}{x}\,dx\right)\right]$$

exists.

Since $\frac{1}{x} \geq \frac{1}{k+1}$ on the interval $[k, k+1]$, it follows that $\int_k^{k+1} \frac{1}{x}\, dx \geq \frac{1}{k+1}$ for each $k \geq 1$, and hence that $\frac{1}{k} - \int_k^{k+1} \frac{1}{x}\, dx \leq \frac{1}{k} - \frac{1}{k+1}$ for each $k \geq 1$. It follows that

$$\sum_{k=1}^{n-1} \left(\frac{1}{k} - \int_k^{k+1} \frac{1}{x}\, dx \right) \leq \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n} \leq 1\,,$$

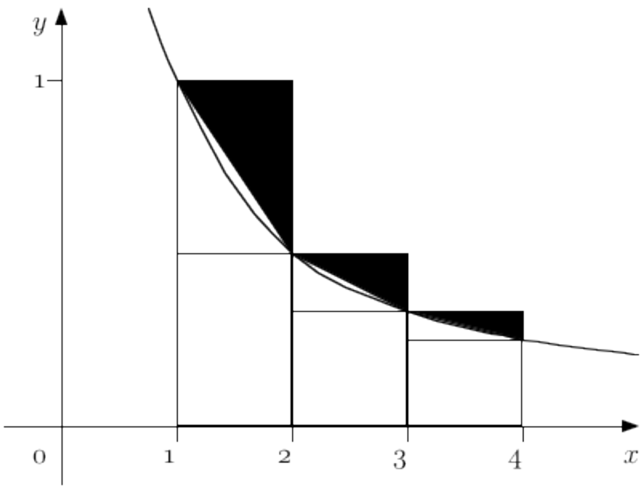
for all $n \geq 1$. (Note that the latter sum telescopes.)

It follows that the desired limit exists, and so γ is actually a well-defined constant. ■

2. Show that $\frac{1}{2} \leq \gamma \leq 1$. *[3]*

Solution. Since, as was noted above, 1 is an upper bound for the terms whose limit is γ, we have $\gamma \leq 1$. It remains to show that $\frac{1}{2} \leq \gamma$.

To see this, note that each area $\frac{1}{k} - \int_k^{k+1} \frac{1}{x}\, dx$ is a little less than that of the triangle with area $\frac{1}{2} \cdot 1 \cdot \left(\frac{1}{k} - \frac{1}{k+1} \right)$, as in the picture below:



It follows that for all $n \geq 1$,

$$\gamma \geq \frac{1}{2}\left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2}\left(1 - \frac{1}{n+1}\right)\,,$$

and so we must have that $\gamma \geq \lim_{n \to \infty} \frac{1}{2}\left(1 - \frac{1}{n+1}\right) = \frac{1}{2}(1 - 0) = \frac{1}{2}$, as desired. ■