Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2009

Solution to Assignment #5

Cauchy sequences

The counterpart for sequences of the Cauchy Convergence Criterion for series is the following notion:

DEFINITION. A sequence a_0, a_1, a_2, ... is a *Cauchy sequence* if for any ε there is an $N \geq 0$ such that if $m > k \geq N$, then $|a_m - a_k| < \varepsilon$.

Note that a series satisfies the Cauchy Convergence Criterion exactly when its partial sums form a Cauchy sequence.

1. Show that a sequence has a limit if and only if it is a Cauchy sequence. *[10]*

SOLUTION. ($\Longrightarrow$) Suppose $\lim_{n\to\infty} a_n = L$ and $\varepsilon > 0$ is given. Then $\frac{\varepsilon}{2} > 0$, and there is some $N \geq 0$ such that for all $k \geq N$, $|a_k - L| < \frac{\varepsilon}{2}$. If $m > k \geq N$, then

$$\begin{aligned} |a_m - a_k| &= |a_m - L + L - a_k| \\ &\leq |a_m - L| + |L - a_k| \\ &= |a_m - L| + |a_k - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,. \end{aligned}$$

It follows that the sequence a_n is a Cauchy sequence. $\square$

($\Longleftarrow$) Suppose that a_n is a Cauchy sequence. Choose N_p for each integer $p \geq 1$ as follows:

Choose N_1 such that for all $m > k \geq N_1$, $|a_m - a_k| < 1$.

Given N_p, choose $N_{p+1} \geq N_p$ such that $m > k \geq N_1$, $|a_m - a_k| < \frac{1}{p}$.

For each integer $p \geq 1$, let $I_p = \left[a_{N_p} - \frac{1}{p}, a_{N_p} + \frac{1}{p}\right]$. Note that it follows that if $n \geq N_p$, then $a_n \in I_p$, and that the width of I_p is $\frac{2}{p}$.

Now consider the intervals $J_p = I_1 \cap I_2 \cap \cdots \cap I_p$, for each $p \geq 1$. Then each J_p a closed interval (being an intersection of closed intervals) which is non-empty, since $a_{N_p} \in I_k$ for $k = 1, \ldots, p$ (this is where we need that $N_1 \leq N_2 \leq \cdots \leq N_p$) and $J_p \supseteq J_{p+1}$ for each $p \geq 1$. By the Nested Interval Principle, it follows that there is some $L \in \bigcap_{p=1}^{\infty} J_p$. Since the width of each J_p is $\leq \frac{2}{p}$ because $J_p \subseteq I_p$, the width of $\bigcap_{p=1}^{\infty} J_p$ is 0, so in fact $\bigcap_{p=1}^{\infty} J_p = \{L\}$.

We claim that $\lim_{n\to\infty} a_n = L$. Suppose $\varepsilon > 0$ is given. Choose an integer $p \geq 1$ such that $\frac{2}{p} < \varepsilon$. Then if $n \geq N = N_p$, $|a_n - L| \leq \frac{2}{p} < \varepsilon$, since both a_n and L are in $I_p = \left[a_{N_p} - \frac{1}{p}, a_{N_p} + \frac{1}{p}\right]$, which has width $\frac{2}{p}$. $\square$

Thus a sequence converges if and only if it is a Cauchy sequence. $\blacksquare$