Mathematics 3790H – Analysis I: Introduction to analysis
Trent University, Fall 2009

Solutions to Assignment #4
A function from heck.

We first need a bit of notation. If x is a real number, let:

$$\begin{aligned}\{x\} &= \text{the distance from } x \text{ to the nearest integer}\\ &= \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}\end{aligned}$$

Note that for any real number x, $0 \le \{x\} \le \frac{1}{2}$. It will be handy later to have a couple of properties of $\{x\}$ in hand.

1. For all $x \in \mathbb{R}$, $\left\{x \pm \frac{1}{2}\right\} = \frac{1}{2} - \{x\}$. *[1]*

Solution. Note that $\{x\} = \frac{1}{2}$ exactly when x is an integer plus (or minus) one half. We break the problem down into cases:

i. If x is an integer, $\{x\} = 0$ and $\left\{x \pm \frac{1}{2}\right\} = \frac{1}{2} = \frac{1}{2} - \{x\}$.

ii. If x is an integer plus (or minus) one half, then $\{x\} = \frac{1}{2}$ and $\left\{x \pm \frac{1}{2}\right\} = 0 = \frac{1}{2} - \{x\}$.

iii. If $n < x < n+\frac{1}{2}$ for some integer n, then $\{x\} = x-n$, while $n+\frac{1}{2} < x+\frac{1}{2} < n+1$ and $n-\frac{1}{2} = (n-1)+\frac{1}{2} < x-\frac{1}{2} < n$. Then $\left\{x+\frac{1}{2}\right\} = n+1-\left(x+\frac{1}{2}\right) = \frac{1}{2}+n-x = \frac{1}{2}-(x-n) = \frac{1}{2}-\{x\}$. Similarly, $\left\{x-\frac{1}{2}\right\} = n-\left(x-\frac{1}{2}\right) = n-x+\frac{1}{2} = \frac{1}{2}-(x-n) = \frac{1}{2}-\{x\}$.

iv. If $n+\frac{1}{2} < x < n+1$ for some integer n, then $\{x\} = (n+1)-x$, while $n+1 < x+\frac{1}{2} < (n+1)+\frac{1}{2}$ and $n < x-\frac{1}{2} < n+\frac{1}{2}$. Then $\left\{x+\frac{1}{2}\right\} = \left(x+\frac{1}{2}\right)-(n+1) = \frac{1}{2}+x-(n+1) = \frac{1}{2}-(x-(n+1)) = \frac{1}{2}-\{x\}$. Similarly, $\left\{x-\frac{1}{2}\right\} = \left(x-\frac{1}{2}\right)-n = \frac{1}{2}+x-1-n = \frac{1}{2}-((n+1)-x) = \frac{1}{2}-\{x\}$. ■

2. For all $x, y \in \mathbb{R}$, $\{x+y\} \le \{x\}+\{y\}$ and $\{x\}-\{y\} \le \{x-y\}$. *[1]*

Solution. We tackle $\{x+y\} \le \{x\}+\{y\}$ first. Note that $x = n+a$ and $y = m+b$ for some integers n and m and real numbers a and b with $|a| = \{x\}$ and $|b| = \{y\}$. Then

$$\begin{aligned}\{x+y\} = \{a+b\} &= \text{distance } a+b \text{ is to the nearest integer}\\ &\le \text{distance } a+b \text{ is from } 0\\ &\le |a|+|b| = \{x\}+\{y\}.\end{aligned}$$

The second inequality follows from the first, using the fact that $x = (x-y)+y$:

$$\begin{aligned}\{x\}-\{y\} \le \{x-y\} &\iff \{x\} \le \{x-y\}+\{y\}\\ &\iff \{x\} = \{(x-y)+y\} \le \{x-y\}+\{y\} \quad ■\end{aligned}$$

We will now define the function we're really interested in. For any real number x, let

$$g(x) = \sum_{n=0}^{\infty} \frac{\{n!x\}}{n!}.$$

One needs to check that this definition really makes sense:

3. Use the Comparison Test (see Chapter 4 in the text) to verify that the series defining $g(x)$ converges no matter what x we pick. *[2]*

SOLUTION. Observe that it follows from **1** that for each $n \geq 0$ and every $x \in \mathbb{R}$,

$$0 \leq \frac{\{n!x\}}{n!} \leq \frac{1/2}{n!} = \frac{1}{2} \cdot \frac{1}{n!} \,.$$

It follows by the Comparison Test that $g(x) = \sum_{n=0}^{\infty} \frac{\{n!x\}}{n!}$ converges for any $x \in \mathbb{R}$ by comparison with $\sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{1}{n!} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1^n}{n!} = \frac{1}{2}e^1 = \frac{e}{2}$. ■

NOTE. One free consequence of this is that $g(x)$ is bounded above by $\frac{e}{2}$.

Note that $g(x) \geq 0$ for all $x \in \mathbb{R}$. [Because $g(x) = \sum_{n=0}^{\infty} \frac{\{n!x\}}{n!}$ is a series of non-negative terms no matter what x is ...] It turns out that $g(x)$ is continuous but not differentiable at every point:

4. Show that $g(x)$ is continuous at $x = a$ for all $a \in \mathbb{R}$. *[4]*

Hint: Given an $\varepsilon > 0$, first choose an N such that $\sum_{n=N+1}^{\infty} \frac{\{n!x\}}{n!} < \frac{\varepsilon}{4}$. (Note that this can be done independently of the value of x ...) Then go to work on $\sum_{n=0}^{N} \frac{\{n!x\}}{n!} - \sum_{n=0}^{N} \frac{\{n!a\}}{n!}$.

SOLUTION. Suppose $a \in \mathbb{R}$ and $\varepsilon > 0$ is given. We need to show that there is a $\delta > 0$ such that if $|x - a| < \delta$, then $|g(x) - g(a)| < \varepsilon$.

Following the hint, choose an N such that $\left| \left(\frac{1}{2} \sum_{n=0}^{N} \frac{1}{n!} \right) - \frac{e}{2} \right| < \frac{\varepsilon}{4}$; such an N exists because $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges to e. Since $\sum_{n=0}^{\infty} \frac{1}{n!}$ is a sequence of positive terms, it follows that $\frac{1}{2} \sum_{n=N+1}^{\infty} \frac{1}{n!} < \frac{\varepsilon}{4}$, and hence, since $\{n!x\} \leq \frac{1}{2}$ for all $x \in \mathbb{R}$, that

$$\sum_{n=N+1}^{\infty} \frac{\{n!x\}}{n!} \leq \sum_{n=N+1}^{\infty} \frac{1}{2} \cdot \frac{1}{n!} = \frac{1}{2} \sum_{n=N+1}^{\infty} \frac{1}{n!} < \frac{\varepsilon}{4} \,.$$

It follows from this that:

$$
\begin{aligned}
|g(x) - g(a)| &= \left| \left(\sum_{n=0}^{\infty} \frac{\{n!x\}}{n!} \right) - \left(\sum_{n=0}^{\infty} \frac{\{n!a\}}{n!} \right) \right| \\
&= \left| \left(\sum_{n=0}^{N} \frac{\{n!x\}}{n!} + \sum_{n=N+1}^{\infty} \frac{\{n!x\}}{n!} \right) - \left(\sum_{n=0}^{N} \frac{\{n!a\}}{n!} + \sum_{n=N+1}^{\infty} \frac{\{n!a\}}{n!} \right) \right| \\
&= \left| \left(\sum_{n=0}^{N} \frac{\{n!x\}}{n!} \right) - \left(\sum_{n=0}^{N} \frac{\{n!a\}}{n!} \right) + \left(\sum_{n=N+1}^{\infty} \frac{\{n!x\}}{n!} \right) - \left(\sum_{n=N+1}^{\infty} \frac{\{n!a\}}{n!} \right) \right| \\
&\leq \left| \left(\sum_{n=0}^{N} \frac{\{n!x\}}{n!} \right) - \left(\sum_{n=0}^{N} \frac{\{n!a\}}{n!} \right) \right| + \left| \sum_{n=N+1}^{\infty} \frac{\{n!x\}}{n!} \right| + \left| \sum_{n=N+1}^{\infty} \frac{\{n!a\}}{n!} \right| \\
&< \left| \left(\sum_{n=0}^{N} \frac{\{n!x\}}{n!} \right) - \left(\sum_{n=0}^{N} \frac{\{n!a\}}{n!} \right) \right| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&= \left| \sum_{n=0}^{N} \frac{\{n!x\} - \{n!a\}}{n!} \right| + \frac{\varepsilon}{2} \leq \left(\sum_{n=0}^{N} \frac{|\{n!x\} - \{n!a\}|}{n!} \right) + \frac{\varepsilon}{2}
\end{aligned}
$$

Obviously, we now need to work on bounding $\sum_{n=0}^{N} \frac{|\{n!x\}-\{n!a\}|}{n!}$. The key to this is the fact that $\{x\}$ is continuous, the proof of which we leave the proof to the reader. (Intuitively, just look at the graph of $\{x\}$.)

Since $\{x\}$ is continuous we can find, for each $n = 0, 1, \dots, N$, a $\delta_n > 0$ such that if $|x - a| < \delta_n$, then $\frac{|\{n!x\}-\{n!a\}|}{n!} < \frac{\varepsilon}{2(N+1)}$. Let $\delta = \min\{\delta_0, \delta_1, \dots, \delta_N\}$. Then $\delta > 0$ and if $|x - a| < \delta \leq \delta_n$ (for each $n = 0, 1, \dots, N$), then

$$|g(x) - g(a)| < \left(\sum_{n=0}^{N} \frac{|\{n!x\} - \{n!a\}|}{n!} \right) + \frac{\varepsilon}{2} < \left(\sum_{n=0}^{N} \frac{\varepsilon}{2(N+1)} \right) + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Hence $g(x)$ is continuous at a for any $a \in \mathbb{R}$. ■

5. Show that $g(x)$ is not differentiable at $x = 0$. *[2]*

Hint: The idea is to construct a sequence $a_n \to 0$ such that $\left| \frac{g(a_n)-g(0)}{a_n-0} \right| \geq 1$ for all n.

Solution. Since nobody completed this one, I'm adding it to the list of bonus assignment problems that came with Assignment #4. ■