Mathematics 3790H – Analysis I: Introduction to analysis
Trent University, Fall 2008

Assignment #3
The integral form of the remainder of a Taylor series

Suppose that a is a real number and $f(x)$ is a function such that $f^{(n)}(x)$ is defined and continuous for all $n \geq 0$ and all values of x we may encounter. The Taylor polynomial of degree n of $f(x)$ at a is defined to be

$$\begin{aligned} T_{n,a}(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \,, \end{aligned}$$

and the corresponding remainder term is

$$f(x) = T_{n,a}(x) + R_{n,a}(x) \,, \ \textit{i.e.}\ R_{n,a}(x) = f(x) - T_{n,a}(x) \,.$$

1. Use the Fundamental Theorem of Calculus to show that

$$R_{0,a}(x) = \int_a^x f'(t)\,dt \,. \quad [1]$$

Solution. By definition $R_{0,a}(x) = f(x) - T_{0,a}(x) = f(x) - f(a)$, and by the Fundamental Theorem of Calculus, $f(x) - f(a) = \int_a^x f'(t)\,dt$. Hence $R_{0,a}(x) = \int_a^x f'(t)\,dt$, as desired. ■

2. Use induction (and some calculus!) to show that

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$$

for $n \geq 0$. (This is the integral form of the remainder of a Taylor series.) *[5]*

Solution. As instructed, we will proceed by induction on n.

Base Step. ($n = 0$) We need to check that $R_{0,a}(x) = \int_a^x \frac{f^{(0+1)}(t)}{0!}(x-t)^0\,dt = \int_a^x f'(t)\,dt$. This is just problem **1**.

Induction Hypothesis. ($n = k$) Assume that$R_{k,a}(x) = \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt$.

Induction Step. ($n = k+1$) By definition and the Induction Hypothesis,

$$\begin{aligned}
R_{k+1,a}(x) = f(x) - T_{k+1,a}(x) &= f(x) - \left[T_{k,a}(x) + \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1}\right] \\
&= [f(x) - T_{k,a}(x)] - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} \\
&= R_{k,a}(x) - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} \\
&= \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1}
\end{aligned}$$

We will now apply integration by parts to the integral $\int_a^x \frac{f^{(k+2)}(t)}{(k+1)!}(x-t)^{k+1}\,dt$. Bearing in mind that t is the variable of integration, the parts are:

$$\begin{aligned}
u &= (x-t)^{k+1} & dv &= \tfrac{f^{(k+2)}(t)}{(k+1)!}\,dt \\
du &= -(k+1)(x-t)^k\,dt & v &= \tfrac{f^{(k+1)}(t)}{(k+1)!}
\end{aligned}$$

Here goes!

$$\begin{aligned}
&\int_a^x \frac{f^{(k+2)}(t)}{(k+1)!}(x-t)^{k+1}\,dt = \int_a^x u\,dv = uv\Big|_a^x - \int_a^x v\,du \\
&= (x-t)^{k+1} \cdot \frac{f^{(k+1)}(t)}{(k+1)!}\Bigg|_a^x - \int_a^x \frac{f^{(k+1)}(t)}{(k+1)!} \cdot (-1)(k+1)(x-t)^k\,dt \\
&= \left[0 - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1}\right] - (-1)\int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt \\
&= \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1}
\end{aligned}$$

Putting the two previous paragraphs together, we get

$$\begin{aligned}
R_{k+1,a}(x) &= \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} \\
&= \int_a^x \frac{f^{(k+2)}(t)}{(k+1)!}(x-t)^{k+1}\,dt\,,
\end{aligned}$$

as desired.

It follows by induction that $R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$ for $n \geq 0$. ■

3. Deduce the Lagrange Remainder Theorem from **2.** *[4]*

Note: For **3** you may assume the *Mean Value Theorem for Integrals*:

> If $f(x)$ is continuous on $[a, b]$ and $g(x)$ is integrable and non-negative (or non-positive) on $[a, b]$, then
>
> $$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$
>
> for some $\xi \in [a, b]$.

SOLUTION. Applying the Mean Value Theorem for integrals to the integral form of $R_{n,a}(x)$ gives us

$$\begin{aligned}
R_{n,a}(x) &= \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt \\
&= \frac{f^{(n+1)}(c)}{n!}\int_a^x (x-t)^n\,dt \quad \text{(for some } c \in [a,b]\text{)} \\
&= \frac{f^{(n+1)}(c)}{n!}\cdot(-1)\frac{(x-t)^{n+1}}{n+1}\bigg|_a^x \\
&= \frac{f^{(n+1)}(c)}{n!}\cdot(-1)\left[0 - \frac{(x-a)^{n+1}}{n+1}\right] \\
&= \frac{f^{(n+1)}(c)}{n!}\cdot\frac{(x-a)^{n+1}}{n+1} \\
&= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},
\end{aligned}$$

which is the Lagrange form of the remainder. We leave it to the reader to figure out why the $c \in [a, b]$ has to be strictly between a and b. ■