Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2008

Assignment #3
The integral form of the remainder of a Taylor series
Due: Thursday, 22 October, 2009

Suppose that a is a real number and $f(x)$ is a function such that $f^{(n)}(x)$ is defined and continuous for all $n \geq 0$ and all values of x we may encounter. The Taylor polynomial of degree n of $f(x)$ at a is defined to be

$$\begin{aligned} T_{n,a}(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \,, \end{aligned}$$

and the corresponding remainder term is

$$f(x) = T_{n,a}(x) + R_{n,a}(x)\,,\ i.e.\ R_{n,a}(x) = f(x) - T_{n,a}(x)\,.$$

1. Use the Fundamental Theorem of Calculus to show that

$$R_{0,a}(x) = \int_a^x f'(t)\,dt\,. \quad [1]$$

2. Use induction (and some calculus!) to show that

$$R_{n,a} = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$$

for $n \geq 0$. (This is the integral form of the remainder of a Taylor series.) *[5]*

3. Deduce the Lagrange Remainder Theorem from **2.** *[4]*

Note: For **3** you may assume the *Mean Value Theorem for Integrals*:

If $f(x)$ is continuous on $[a, b]$ and $g(x)$ is integrable and non-negative (or non-positive) on $[a, b]$, then

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$

for some $\xi \in [a, b]$.