Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2008

Solutions to Assignment #3
Eeeeeeeeeeeeeeeeee!

Recall that the Taylor series at 0 of e^x is $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. Let $R_{n,0}(x)$ denote the nth remainder term at 0, *i.e.*

$$\begin{aligned} R_{n,0}(x) &= e^x - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) \\ &= e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\,. \end{aligned}$$

1. Use the integral form of the remainder for a Taylor series (see Assignment #2) to show that for $x > 0$, $0 < R_{n,0}(x) \leq \dfrac{e^x x^{n+1}}{(n+1)!}$. *[2]*

Solution. From Assignment #2 we know that for all $n \geq 0$,

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt\,.$$

In our case, $a = 0$ and $f(t) = e^t$. Since $\frac{d}{dt}e^t = e^t$, it follows that $f^{(n+1)}(t) = e^t$ for all $n \geq 0$, which means that in this case we have, for $n \geq 0$,

$$R_{n,0}(x) = \int_0^x \frac{e^t}{n!}(x-t)^n\,dt\,.$$

Since $x > 0$ and e^t is a continuous increasing function, we know that for all t with $0 \leq t \leq x$ we have that $e^t \leq e^x$. Hence

$$\begin{aligned} R_{n,0}(x) = \int_0^x \frac{e^t}{n!}(x-t)^n\,dt &\leq \int_0^x \frac{e^x}{n!}(x-t)^n\,dt \\ &= \frac{e^x}{n!}\int_0^x (x-t)^n\,dt \\ &= \frac{e^x}{n!}\,(-1)\frac{(x-t)^{n+1}}{n+1}\bigg|_0^x \\ &= \frac{e^x}{n!}\left((-1)\frac{(x-x)^{n+1}}{n+1} - (-1)\frac{(x-0)^{n+1}}{n+1}\right) \\ &= \frac{e^x}{n!}\cdot\frac{x^{n+1}}{n+1} \\ &= \frac{e^x x^{n+1}}{(n+1)!}\,, \end{aligned}$$

as desired. ■

2. Use your estimate for $R_{n,0}(x)$ in **1** to show that $0 < R_{n,0}(1) < \dfrac{3}{(n+1)!}$. *[1]*

Solution. From the estimate in **1**, with $x = 1$:

$$0 < R_{n,0}(1) \leq \frac{e^1 1^{n+1}}{(n+1)!} = \frac{e}{(n+1)!}$$

Since $e = 2.71 \cdots < 3$, it follows that

$$0 < R_{n,0}(1) \leq \frac{e}{(n+1)!} < \frac{3}{(n+1)!},$$

as desired. ■

3. Show that e is irrational. *[7]*

Hint: Assume by way of contradiction that $e = \frac{a}{b}$, where a and b are positive integers. Choose an n such that $n > 3$ and $n > b$, and use the fact that

$$\frac{a}{b} = e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_{n,0}(1)$$

to show that $n!R_{n,0}(1)$ must be an integer. Then use **2** to show that it cannot be an integer.

Solution. Following the hint, assume by way of contradiction that e is rational, *i.e.* $e = \frac{a}{b}$, where a and b are positive integers.

Let n be an integer such that $n > 3$ and $n > b$. By the definition of $R_{n,0}(1)$ we have that:

$$\frac{a}{b} = e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_{n,0}(1)$$

Multiplying through by $n!$ gives us:

$$\frac{n!a}{b} = n! + \frac{n!}{1!} + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + \frac{n!}{n!} + n!R_{n,0}(1)$$

Since $n > b$, b is a factor of $n!$, and so $\frac{n!a}{b}$ must be an integer. It is easy to see from the definition of the factorial function that each of $n!$, $\frac{n!}{1!}$, $\frac{n!}{2!}$, $\frac{n!}{3!}$, $\ldots$, $\frac{n!}{n!}$, must be an integer too. It follows that

$$n!R_{n,0}(1) = \frac{n!a}{b} - \left(n! + \frac{n!}{1!} + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + \frac{n!}{n!}\right)$$

must be an integer too.

On the other hand, we know from **2** that $0 < R_{n,0}(1) < \frac{3}{(n+1)!}$, so $0 < n!R_{n,0}(1) < \frac{3 \cdot n!}{(n+1)!} = \frac{3}{n+1}$. Since $n > 3$, it is easy to see that $\frac{3}{n+1} < \frac{3}{4} < 1$. It follows that $0 < n!R_{n,0}(1) < 1$, which contradicts the fact that $n!R_{n,0}(1)$ must be an integer.

Hence, by contradiction, e must be irrational. ■