

Mathematics 3790H – Analysis I: Introduction to analysis

TRENT UNIVERSITY, Fall 2008

Solutions to Assignment #1

Recall that our objective was to show that  $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}$ .

1. Verify the following trigonometric identity. (So long as  $x$  is not an integer multiple of  $\pi$  anyway!) [1.5]

$$\frac{1}{\sin^2(x)} = \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{x}{2}\right)} + \frac{1}{\sin^2\left(\frac{x+\pi}{2}\right)} \right]$$

*Hint:* Use common trig identities and the fact that for any  $t$ ,  $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$ .

**Solution.** Here goes!

$$\begin{aligned} \frac{1}{\sin^2(x)} &= \frac{1}{(\sin(x))^2} = \frac{1}{(2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right))^2} = \frac{1}{4 \sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right)} \\ &= \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right)} \right] = \frac{1}{4} \left[ \frac{\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right)} \right] \\ &= \frac{1}{4} \left[ \frac{\sin^2\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right)} + \frac{\cos^2\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right)} \right] = \frac{1}{4} \left[ \frac{1}{\cos^2\left(\frac{x}{2}\right)} + \frac{1}{\sin^2\left(\frac{x}{2}\right)} \right] \\ &= \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{x}{2} + \frac{\pi}{2}\right)} + \frac{1}{\sin^2\left(\frac{x}{2}\right)} \right] = \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{x}{2}\right)} + \frac{1}{\sin^2\left(\frac{x+\pi}{2}\right)} \right] \quad \blacksquare \end{aligned}$$

2. Verify the following trigonometric summation formula for  $m \geq 1$ . [1.5]

$$1 = \frac{2}{4^m} \sum_{k=0}^{2^{m-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{m+1}}\right)}$$

*Hint:* Apply the identity from question 1 repeatedly, starting from  $1 = \frac{1}{\sin^2\left(\frac{\pi}{2}\right)}$ . After doing so, you may find the fact that  $\sin(t) = \sin(\pi - t)$  comes in handy.

**Solution.** Note that  $\sin\left(\frac{\pi}{2}\right) = 1$ , so  $1 = \frac{1}{1^2} = \frac{1}{\sin^2\left(\frac{\pi}{2}\right)}$ .

First, using the identity in 1 with  $x = \frac{\pi}{2}$ , we get:

$$1 = \frac{1}{\sin^2\left(\frac{\pi}{2}\right)} = \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{\pi/2}{2}\right)} + \frac{1}{\sin^2\left(\frac{\pi/2+\pi}{2}\right)} \right] = \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{\pi}{4}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{4}\right)} \right]$$

Second, using the identity in **1** on each part of the formula above, with  $x = \frac{\pi}{4}$  and  $x = \frac{3\pi}{4}$  respectively, we get (omitting some of the algebra and arithmetic):

$$\begin{aligned}
1 &= \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{\pi}{4}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{4}\right)} \right] \\
&= \frac{1}{4} \left( \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{\pi/2+\pi}{8}\right)} \right] + \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{3\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{\pi/2+3\pi}{8}\right)} \right] \right) \\
&= \frac{1}{4} \left( \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{5\pi}{8}\right)} \right] + \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{3\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{7\pi}{8}\right)} \right] \right) \\
&= \frac{1}{16} \left[ \frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{5\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{7\pi}{8}\right)} \right] \\
&= \frac{1}{4^2} \sum_{k=0}^{2^{2+1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{2+1}}\right)}
\end{aligned}$$

If we were to continue, at the  $n$ th step we would start with the formula from step  $n - 1$ , namely:

$$1 = \frac{1}{4^n} \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^n}\right)}$$

Next, we'd use the identity in **1** on each part of this formula, with  $x = \frac{\pi}{2^n}$ ,  $x = \frac{3\pi}{2^n}$ ,  $\dots$ , and  $x = \frac{(2^n-1)\pi}{2^n}$  respectively. Applied to  $x = \frac{(2k+1)\pi}{2^n}$ , this will give:

$$\begin{aligned}
\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^n}\right)} &= \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{(2k+1)\pi/2^n}{2}\right)} + \frac{1}{\sin^2\left(\frac{(2k+1)\pi/2^n+\pi}{2}\right)} \right] \\
&= \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \frac{1}{\sin^2\left(\frac{(2k+1)\pi+2^n\pi}{2^{n+1}}\right)} \right] \\
&= \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \frac{1}{\sin^2\left(\frac{(2(k+2^{n-1})+1)\pi}{2^{n+1}}\right)} \right]
\end{aligned}$$

It follows that

$$\begin{aligned}
1 &= \frac{1}{4^{n-1}} \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^n}\right)} \\
&= \frac{1}{4^{n-1}} \sum_{k=0}^{2^{n-1}-1} \frac{1}{4} \left[ \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \frac{1}{\sin^2\left(\frac{(2(k+2^{n-1})+1)\pi}{2^{n+1}}\right)} \right] \\
&= \frac{1}{4^n} \sum_{k=0}^{2^{n-1}-1} \left[ \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \frac{1}{\sin^2\left(\frac{(2(k+2^{n-1})+1)\pi}{2^{n+1}}\right)} \right] \\
&= \frac{1}{4^n} \left[ \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2(k+2^{n-1})+1)\pi}{2^{n+1}}\right)} \right] \\
&= \frac{1}{4^n} \left[ \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \sum_{k=2^{n-1}}^{2^n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} \right] \\
&= \frac{1}{4^n} \sum_{k=0}^{2^n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)}
\end{aligned}$$

A really formal argument would proceed by induction on  $n$ .

Unfortunately, what we have so far isn't quite the formula that was asked for. One step more is needed to get there, the key to which is the observation that  $\sin(x) = \sin(\pi - x)^*$ . In particular, for  $0 \leq k \leq 2^n - 1$ ,

$$\pi - \frac{(2k+1)\pi}{2^{n+1}} = \frac{2^{n+1} - 2k - 1}{2^{n+1}}\pi = \frac{2(2^n - k) - 1}{2^{n+1}}\pi = \frac{2(\ell+1) - 1}{2^{n+1}}\pi,$$

where  $0 \leq \ell < 2^n - 1$  and  $\ell = 2^n - k - 1$ . It follows that the terms in the second half of the sum  $\frac{1}{4^n} \sum_{k=0}^{2^n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)}$  repeat the values of the terms in the first half, albeit in reverse order. We can now finish the job:

$$\begin{aligned}
1 &= \frac{1}{4^n} \sum_{k=0}^{2^n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} = \frac{1}{4^n} \left[ \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \sum_{k=2^{n-1}}^{2^n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} \right] \\
&= \frac{1}{4^n} \left[ \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \sum_{\ell=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2\ell+1)\pi}{2^{n+1}}\right)} \right] = \frac{2}{4^n} \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} \quad \blacksquare
\end{aligned}$$

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\* You can get this from the addition formula for sin:

$$\sin(\pi - x) = \sin(\pi) \cos(-x) + \cos(\pi) \sin(-x) = 0 \cdot \cos(-x) + (-1)(-\sin(x)) = \sin(x)$$

3. Verify the following limit formula, where  $k \geq 0$  is fixed. [2]

$$\lim_{m \rightarrow \infty} 2^m \sin \left( \frac{(2k+1)\pi}{2^{m+1}} \right) = \frac{(2k+1)\pi}{2}$$

*Hint:* This is really just a version of  $\lim_{t \rightarrow 0} \frac{\sin(t)}{t} = 1$ , which limit you may remember from MATH 110 ...

**Solution.** Recall that  $\lim_{t \rightarrow 0} \frac{\sin(t)}{t} = 1$ .

$$\begin{aligned} \lim_{m \rightarrow \infty} 2^m \sin \left( \frac{(2k+1)\pi}{2^{m+1}} \right) &= \lim_{m \rightarrow \infty} \frac{\frac{(2k+1)\pi}{2}}{\frac{(2k+1)\pi}{2}} 2^m \sin \left( \frac{(2k+1)\pi}{2^{m+1}} \right) \\ &= \frac{(2k+1)\pi}{2} \lim_{m \rightarrow \infty} \frac{2^m}{\frac{(2k+1)\pi}{2}} \sin \left( \frac{(2k+1)\pi}{2^{m+1}} \right) \\ &= \frac{(2k+1)\pi}{2} \lim_{m \rightarrow \infty} \frac{\sin \left( \frac{(2k+1)\pi}{2^{m+1}} \right)}{\frac{(2k+1)\pi}{2^{m+1}}} \\ &= \frac{(2k+1)\pi}{2} \lim_{t \rightarrow 0} \frac{\sin(t)}{t} = \frac{(2k+1)\pi}{2} \cdot 1 = \frac{(2k+1)\pi}{2} \quad \blacksquare \end{aligned}$$

4. Take the limit as  $m \rightarrow \infty$  of the identity in 2, and use 3 to show the following. [2]

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

**Solution.** Here goes!

$$\begin{aligned} 1 &= \lim_{m \rightarrow \infty} 1 = \lim_{m \rightarrow \infty} \frac{2}{4^m} \sum_{k=0}^{2^{m-1}-1} \frac{1}{\sin^2 \left( \frac{(2k+1)\pi}{2^{m+1}} \right)} = \lim_{m \rightarrow \infty} \sum_{k=0}^{2^{m-1}-1} \frac{2}{4^m \sin^2 \left( \frac{(2k+1)\pi}{2^{m+1}} \right)} \\ &= \lim_{m \rightarrow \infty} \sum_{k=0}^{2^{m-1}-1} \frac{2}{2^{2m} \sin^2 \left( \frac{(2k+1)\pi}{2^{m+1}} \right)} = \lim_{m \rightarrow \infty} \sum_{k=0}^{2^{m-1}-1} \frac{2}{\left[ 2^m \sin \left( \frac{(2k+1)\pi}{2^{m+1}} \right) \right]^2} \\ &= \sum_{k=0}^{\infty} \lim_{m \rightarrow \infty} \frac{2}{\left[ 2^m \sin \left( \frac{(2k+1)\pi}{2^{m+1}} \right) \right]^2} = \sum_{k=0}^{\infty} \frac{2}{\left[ \lim_{m \rightarrow \infty} 2^m \sin \left( \frac{(2k+1)\pi}{2^{m+1}} \right) \right]^2} \\ &= \sum_{k=0}^{\infty} \frac{2}{\left[ \lim_{m \rightarrow \infty} 2^m \sin \left( \frac{(2k+1)\pi}{2^{m+1}} \right) \right]^2} = \sum_{k=0}^{\infty} \frac{2}{\left[ \frac{(2k+1)\pi}{2} \right]^2} \\ &= \sum_{k=0}^{\infty} \frac{2}{\frac{(2k+1)^2 \pi^2}{2^2}} = \sum_{k=0}^{\infty} \frac{8}{(2k+1)^2 \pi^2} = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \end{aligned}$$

Multiplying through by  $\frac{\pi^2}{8}$ , it follows that  $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$ . ■

5. Use 4 and some algebra to check that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

is true. [1.5]

*Hint:* Split up  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  into the sums of the terms for even and odd  $n$  respectively and try to rewrite the sum of the terms for even  $n$ .

**Solution.** First, following up on the hint:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \left[ \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \cdots \right] + \left[ \frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots \right] \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \sum_{k=0}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{8} + \sum_{k=0}^{\infty} \frac{1}{4k^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{k^2} \end{aligned}$$

It follows that  $\left(1 - \frac{1}{4}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$ , so  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6}$ . ■

6. A major assumption needs to be made in one of the steps outlined above. What is it? How can it be justified? [1.5]

**Solution.** The tricky part is in 4. Taking the limit of the partial sums at the same time as one takes the limit of the individual terms, *i.e.* the step

$$\lim_{m \rightarrow \infty} \sum_{k=0}^{2^{m-1}-1} \frac{2}{\left[2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)\right]^2} = \sum_{k=0}^{\infty} \lim_{m \rightarrow \infty} \frac{2}{\left[2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)\right]^2}$$

in the solution above, needs some justification. This procedure can be justified, but it takes either an appeal to a fairly sophisticated result (*e.g.* Tannery's Theorem) or some *ad hoc* argument that it safe to do this in this case. ■

REFERENCE (This is where I stole the argument!)

1. Josef Hofbauer, *A Simple Proof of  $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$  and Related Identities*, *American Mathematical Monthly*, Volume 109, Number 2, February 2002, pp. 196–200.