Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2008

Assignment #3
Eeeeeeeeeeeeeeeeee!
Due: Friday, 17 October, 2008

Recall that the Taylor series at 0 of e^x is $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. Let $R_{n,0}(x)$ denote the nth remainder term at 0, *i.e.*

$$\begin{aligned} R_{n,0}(x) &= e^x - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) \\ &= e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \,. \end{aligned}$$

1. Use the integral form of the remainder for a Taylor series (see Assignment #2) to show that for $x > 0$, $0 < R_{n,0}(x) \leq \dfrac{e^x x^{n+1}}{(n+1)!}$. *[2]*

2. Use your estimate for $R_{n,0}(x)$ in **1** to show that $0 < R_{n,0}(1) < \dfrac{3}{(n+1)!}$. *[1]*

3. Show that e is irrational. *[7]*

Hint: Assume by way of contradiction that $e = \frac{a}{b}$, where a and b are positive integers. Choose an n such that $n > 3$ and $n > b$, and use the fact that

$$\frac{a}{b} = e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_{n,0}(1)$$

to show that $n!R_{n,0}(1)$ must be an integer. Then use **2** to show that it cannot be an integer.