Mathematics 3770H – Complex Analysis
TRENT UNIVERSITY, Winter 2024

Take-Home Final Examination
*Due on Thursday, 18 April.**

Instructions: Do Parts **I** and **II** and, if you wish, Part **III** as well. Give complete answers to receive full credit, *a.k.a. show you work!* You may use your texts from this and any other courses, as well as any handouts, class notes, and the like; you may also ask the instructor to clarify the instructions or any of the questions; and you may use a calculator or computer to perform any necessary calculations. *You may not consult any other sources, nor give or receive any other aid on this exam, except with the intructor's explicit permission or as otherwise indicated on a given problem.*

Part I – Tidbits. Do any *ten* (10) of questions **1**–**12**. *[50 = 10×5 each.]*

1. Find all $z \in \mathbb{C}$ such that $z^2 + z + 1 = 0$. *[5]*

2. Suppose γ is any piecewise smooth path from 0 to $\pi + 2i$. Evaluate $\displaystyle\int_\gamma \cos\left(\frac{z}{2}\right)\, dz$. *[5]*

3. Find all $z \in \mathbb{C}$ such that $\sin(z) = 0$. *[5]*

4. Evaluate $\displaystyle\int_\eta \frac{e^{2z}}{z^2 - 4}\, dz$, where η is the circle of radius 1 with centre 2, oriented counterclockwise. *[5]*

5. Suppose $p(z)$ is a polynomial and α is any simple piecewise smooth closed curve in $\mathbb{C}$. Explain why $\displaystyle\int_\alpha p(z)\, dz = 0$. *[5]*

6. For what $a \in \mathbb{C}$ does $e^z = a$ have a solution $z \in \mathbb{C}$. Explain why. *[5]*

7. Write $a = (1+i)\left(1+i\sqrt{3}\right)$ in polar form. *[5]*

8. For what values of $z \in \mathbb{C}$ does the series $\displaystyle\sum_{n=41}^{\infty} \frac{z^n}{n^2}$ converge? *[5]*

9. Find the fixed points of $f(z) = \dfrac{1+z}{1-z}$, *i.e.* the $z \in \mathbb{C}$ such that $f(z) = z$. *[5]*

10. For the multiple-valued logarithm function, is there a difference between the set of all values of $\log\left(z^2\right)$ and the set of all values of $2\log(z)$? Explain why or why not. *[5]*

11. Find an example of two harmonic functions whose product is not harmnic. *[5]*

12. For what values of $z \in \mathbb{C}$ does the series $\displaystyle\sum_{n=41}^{\infty} \frac{n^2}{z^n}$ converge? *[5]*

*Parts **II** and **III** are on page 2.*

* You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission via Blackboard fails, please submit your work to your instructor by email or on paper.

Part II. Do any *five* (5) of questions **13**–**20**. *[50 = 5×10 each.]*

13. Suppose $f(z)$ is a harmonic function on $\mathbb{C}$ and for some $a \in \mathbb{C}$, $|f(z)| \leq |f(a)|$ for all $z \in \mathbb{C}$. Show that $f(z)$ is constant. *[10]*

14. Suppose G is a region in $\mathbb{C}$ and the sequence of functions $f_n : G \to C$ converges uniformly on G to the zero function, *i.e.* $f_n \overset{\text{unif}}{\longrightarrow} 0$. Show that if $\{a_n\}$ is any sequence whatsoever from G, then $\lim_{n\to\infty} f_n(a_n) = 0$. *[10]*

15. Find the Laurent series at $a = 0$ of $f(z) = \dfrac{1}{z^2(1-z)}$ and determine its annulus of convergence. *[10]*

16. Determine, with proof, whether $f(z) = \begin{cases} \bar{z}^2/z & z \neq 0 \\ 0 & z = 0 \end{cases}$ is differentiable at $z = 0$. *[10]*

17. Suppose that the coefficients of the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ are bounded, *i.e.* there is some $M \in \mathbb{R}$ such that $|c_n| \leq M$ for all $n \geq 0$. Show that the radius of convergence of the power series is at least 1. *[10]*

18. Suppose $g(z) = \dfrac{az+b}{cz+d}$ is a Möbius transformation for which $c \neq 0$ and $d \neq 0$. Find the Taylor series at 0 for $g(z)$ and determine its radius of convergence. *[10]*

19. Find the maximum and minimum values of $\left|z^2 - 4\right|$ for z in the closed unit disk, *i.e.* for z such that $|z| \leq 1$. *[10]*

20. Use the Residue Theorem to evaluate $\int_\beta e^{1/z^2}\, dz$, where β is the circle of radius 2 with centre 0, oriented counterclockwise. *[10]*

[Total = 100]

Part III. *Bonus time!* Do neither, either, or both of questions **41** and **42**.

41. Write an original poem touching on complex analysis or mathematics in general. *[1]*

42. When does $6 \times 9 = 42$? *[1]*

I HOPE THAT YOU ENJOYED THE COURSE.
ENJOY THE SUMMER!