**Mathematics 3770H – Complex Analysis**
TRENT UNIVERSITY, Winter 2024

**Assignment #1**
**A Little Matrix Algebra Warm-Up**

*Due[*] just before midnight on Friday, 19 January.*

Recall that the complex numbers are basically the real numbers with a square root for $-1$, usually denoted by $i$, thrown in and then closed up under the usual arithmetic operations of addition and multiplication. A little more formally, the set of complex numbers is

$$\mathbb{C} = \{\, a + bi \mid a,\, b \in \mathbb{R}\,\},$$

with $+$ and $\cdot$ defined by

$$\begin{aligned} (a+bi) + (c+di) &= (a+c) + (b+d)i \\ \text{and } (a+bi)\cdot(c+di) &= (ac-bd) + (ad+bc)i. \end{aligned}$$

Note that his definition of multuplication gives us $i^2 = (0+1i)^2 = -1 + 0i = -1$. Also, note that $\mathbb{R} = \{\, a + bi \in \mathbb{C} \mid b = 0\,\}$ is a subset of $\mathbb{C}$.

Let $\mathbf{M}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a,\, b,\, c,\, d \in \mathbb{R} \right\}$ be the set of $2 \times 2$ matrices with entries from the real numbers, and let $\mathbf{O}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the $2 \times 2$ zero and identity matrices, respectively, in $\mathbf{M}_2(\mathbb{R})$.

There is a copy of $\mathbb{C}$ in $\mathbf{M}_2(\mathbb{R})$:

**1.** Find a matrix $\mathbf{T} \in \mathbf{M}_2(\mathbb{R})$ such that $\mathbf{T}^2 = -\mathbf{I}_2$. *[1]*

Given a matrix $\mathbf{T}$ such that $\mathbf{T}^2 = -\mathbf{I}_2$, we can define a function $\varphi : \mathbb{C} \to \mathbf{M}_2(\mathbb{R})$ by $\varphi(a + bi) = a\mathbf{I}_2 + b\mathbf{T}$.

**2.** Show that the function $\varphi : \mathbb{C} \to \mathbf{M}_2(\mathbb{R})$ is 1–1, and preserves both addition and multiplication,. (That is, for all $z,\, w \in \mathbb{C}$, we have $\varphi(z + w) = \varphi(z) + \varphi(w)$ and $\varphi(z \cdot w) = \varphi(z) \cdot \varphi(w)$.) *[6]*

The next step beyond the complex numbers are the *quaternions*, usually denoted by $\mathbb{H}$. They were invented/discovered in 1843 by William Rowan Hamilton (1805-1865), who used them to do things we mostly do with cross-products nowadays. To make the quaternions, you throw three different square roots of $-1$ – usually denoted by $i$, $j$, and $k$ – into the real numbers which have a non-commutative mutliplication among themselves. To be precise, we have:

$$\begin{array}{ccc} i^2 = -1 & j^2 = -1 & k^2 = -1 \\ ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

[*] You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission via Blackboard fails, please submit your work to your instructor by email or on paper.

Let $\mathbf{M}_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a,\, b,\, c,\, d \in \mathbb{C} \right\}$ be the set of $2 \times 2$ matrices with entries from the complex numbers. As in $\mathbf{M}_2(\mathbb{R})$, let $\mathbf{O}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the $2 \times 2$ zero and identity matrices, respectively, in $\mathbf{M}_2(\mathbb{C})$.

**3.** Find matrices $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W} \in \mathbf{M}_2(\mathbb{C})$ such that:

$$\begin{array}{ccc} \mathbf{U}^2 = -\mathbf{I}_2 & \mathbf{V}^2 = -\mathbf{I}_2 & \mathbf{W}^2 = -\mathbf{I}_2 \\ \mathbf{UV} = \mathbf{W} & \mathbf{VW} = \mathbf{U} & \mathbf{WU} = \mathbf{V} \\ \mathbf{VU} = -\mathbf{W} & \mathbf{WV} = -\mathbf{U} & \mathbf{UW} = -\mathbf{V} \end{array} \quad \textit{[3]}$$

One could go on to use these matrices to show that there is a copy of the quaternions in $\mathbf{M}_2(\mathbb{C})$, but we'll save that as a possibility for another day. :-)