Mathematics 3770H – Complex Analysis
TRENT UNIVERSITY, Winter 2022

Take-home Final Examination
Due on Friday, 22 April.

Instructions: Do Parts **I** – **III** and, if you wish, Part **IIII** as well. Give complete answers to receive full credit. You may use your texts from this and any other courses, as well as any handouts, class notes, and the like; you may also ask the instructor to clarify the instructions or any of the questions; and you may use a calculator or computer to perform any necessary calculations. *You may not consult any other sources, nor give or receive any other aid on this exam, except with the intructor's explicit permission or as otherwise indicated on a given problem.*

Part I. Do all three of **1**–**3**.

1. Do all three of **a**–**c**.

a. Write $(1-i)^6$ both in polar form $re^{i\theta}$ and standard form $a+bi$. *[3]*

b. Let $f(z) = z^3$. Find all the fixed points of $f(z)$, *i.e.* all the points $z \in \mathbb{C}$ such that $f(z) = z$. *[3]*

c. Graph the set $C = \left\{ z \in \mathbb{C} \mid 4\left[z - (\pi + ei)\right]^{-1} = \bar{z} - \pi + ei \right\}$ in the complex plane, and explain just what this set is. *[4]*

2. Do both of **a** and **b**.

a. Suppose $a, b, c \in \mathbb{C}$ are constants. Use the definition of limit to show that $\lim_{z \to b} (az + c) = ab + c$, where z is a complex variable. *[5]*

b. Define $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $g(x+yi) = \dfrac{x - iy}{x^2 + y^2}$. Determine for which $z = x + iy$ in $\mathbb{C} \setminus \{0\}$ this function is differentiable. *[5]*

3. Suppose $h : \mathbb{C} \to \mathbb{C}$ is a function such that both $h(z)$ and $\overline{h(z)}$ are holomorphic on all of $\mathbb{C}$. Show that $h(z)$ is a constant function. *[10]*

Part II. Do any *two* of **4**–**6**.

4. Suppose $m(z)$ is a Möbius transformation that is not equal to the identity function. Show that $m(z)$ has at most two fixed points, *i.e.* there are at most two $z \in \mathbb{C}$ such that $m(z) = z$. *[10]*

5. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_{C_z} \dfrac{w^2 + zw + 1}{z^2 + w^2}\, dw$, where C_z is the unit circle centred at z, oriented positively. Find the simplest expression that you can that is equal to $f(z)$ for all $z \in \mathbb{C}$. *[10]*

6. Show that if $h : \mathbb{C} \to \mathbb{C}$ is entire, *i.e.* holomorphic on all of $\mathbb{C}$, and there exists a real number $L > 0$ such that $|h(z)| \geq L$ for all $z \in \mathbb{C}$, then $h(z)$ is constant. *[10]*

Parts III and IIII are on page 2.

Parts I and II are on page 1.

Part III. Do any *two* of **7**–**9**.

7. Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ is given by $u(x,y) = e^x \sin(y)$. Show that $u(x,y)$ is a harmonic function on its entire domain and find an entire function, *i.e.* one holomorphic on all of $\mathbb{C}$, whose real part is u. *[10]*

8. Suppose $a \in \mathbb{C}$ and r is a real number such that $r > |a|$. Show that $\sum_{n=0}^{\infty} \frac{a^n}{z^n}$ converges uniformly on $A = \{\, z \in \mathbb{C} \mid |z| \geq r \,\}$. *[10]*

9. Find the Laurent series centred at $z = 2$ of $f(z) = \frac{1}{z(z-2)^2}$ and determine the region in which it converges. *[10]*

[Total = 70]

Part IIII. Bonus!

10. Write an original poem touching on complex analysis or mathematics in general. *[1]*

I hope that you enjoyed the course.
Even if you didn't, enjoy the summer!