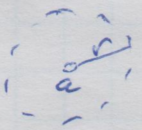
Singularities & Residues

<u>Def'n:</u> The <u>punctured</u> <u>disk</u> of radius r and with centre $a \in \mathbb{C}$ is the set



$$D = \{ z \in \mathbb{C} \mid 0 < |z-a| < r \}.$$

[ie this is just the open disk of radius r around a with a itself removed]

(Mam)

<u>Def'n:</u> If $f(z)$ is holomorphic in some punctured disk with centre $a \in \mathbb{C}$ and $r > 0$, but is not differentiable at a, then a is an (<u>isolated</u>) <u>singularity</u> of $f(z)$. This singularity is said to be:

i) <u>removable</u> if there is a function $g(z)$ holomorphic in $\{z \mid |z-a| < r\}$ [so $g(z)$ is diff'ble at a] s.t. $g(z) = f(z)$ for all $z \in \mathbb{C}$ with $0 < |z-a| < r$.

ii) a <u>pole</u> if $\lim_{z \to a} |f(z)| = \infty$.

iii) <u>essential</u> if it is neither removable, nor a pole.

Examples:

1° $f(z) = \frac{1}{z}$ has a pole at 0: easy to check.

2° $g(z) = \frac{\sinh(z)}{z}$ has a removable singularity at 0: Obviously undefined at 0...

On the other hand,

$$g(z) = \frac{\sinh(z)}{z} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad \left(\text{Since } \sinh(z) = \frac{e^z - e^{-z}}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+1)!}$$ which cvgs. for all z & is diff'ble at 0 (and all other $z \in \mathbb{C}$).

3° $h(z) = e^{1/z}$ has an essential singularity at 0: Obviously undefined at 0...

Also, $\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty$

and $\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$,

so $h(z)$ cannot have a removable discontinuity (much less singularity) at 0 by the first limit, nor a pole at 0, by the second limit.

Some facts about singularities:

<u>Thm:</u> Suppose $a \in \mathbb{C}$ is an isolated singularity of $f(z)$.

Then 1) a is a removable singularity of $f(z)$

$$\Leftrightarrow \lim_{z \to a} (z-a) f(z) = 0 \text{; and}$$

2) a is a pole of $f(z)$

$$\Leftrightarrow \lim_{z \to a} (z-a)^{n+1} f(z) = 0 \text{ for some}$$

$n > 0$. [The least such n is the <u>order</u> of the pole at a.]

<u>Thm:</u> Suppose $a \in \mathbb{C}$ is an isolated singularity of $f(z)$, and $f(z)$ is <u>equal</u> to its Laurent series at a in some punctured disk centred at a,

<u>ie</u> $$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \text{ in that disk.}$$

Then 1) a is a removable ~~[illegible]~~ singularity of $f(z) \Leftrightarrow c_n = 0$ for all $n < 0$,

and 2) a is a pole of $f(z)$ <u>iff</u> there are only finitely many $n < 0$ s.t. $c_n \neq 0$, and it is of order $|n|$ for the least $n < 0$ s.t. $c_n \neq 0$,

and 3) a is an essential singularity of $f(z)$ <u>iff</u> $c_n \neq 0$ for infinitely many $n < 0$.

Of particular usefulness are the coefficients for $n=-1$ in the Laurent series.

<u>Def'n:</u> Suppose a is an isolated singularity of $f(z)$ and $\sum_{n=-\infty}^{\infty} c_n(z-a)^n$ is the Laurent series of $f(z)$ at a. Then $c_{-1} = \mathrm{Res}(f(z),a)$ is the <u>residue of $f(z)$ at a</u>.

<u>Residue Theorem:</u> Suppose $f(z)$ is holomorphic in the region G, except for isolated singularities, and γ is a positively oriented, piecewise smooth, simple, and closed curve in G that does not pass through any of the singularities of $f(z)$ and such that $\gamma \sim_G 0$. Then

1) there are only finitely many singularities of $f(z)$ inside γ,

and 2) $$\int_\gamma f(z)\,dz = 2\pi i \sum_{\substack{a \text{ is a} \\ \text{singularity} \\ \text{inside } \gamma}} \mathrm{Res}(f(z),a)$$