Thm: Suppose $f(z)$ is holomorphic in the annulus $A = \{z \in \mathbb{C} \mid T < |z-a| < U\}$ and $T < r < U$. Let C be the (positively oriented) circle of radius r centred at a. Then

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \quad \text{where } c_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^{n+1}}\,dw,$$

for all z in the annulus A.

proof: Without loss of generality we may assume that $a = 0$. [Otherwise, replace $f(z)$ by $g(z) = f(z+a)$.]

Choose r_1, r_2 s.t. $T < r_1 < r_2 < U$ and suppose C_1 (−ve orientation) & C_2 (+ orientation) are the circles of radius r_1 & r_2, resp., centred at 0, and suppose $r_1 < |z| < r_2$. By the Lemma proved last time,

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw.$$

Now $$\frac{1}{w-z} = \frac{1}{w}\cdot\frac{1}{1-\frac{z}{w}} = \frac{1}{w}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n$$

For $w \in C_2$ we have $|w| = r_2 > |z|$, so $\left|\frac{z}{w}\right| < 1$, and so the series converges (uniformly) in w.

Thus $$\frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dz = \frac{1}{2\pi i}\int_{C_2}\left[f(w)\cdot\frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n\right]dw$$

$$= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w}\cdot\left(\frac{z}{w}\right)^n dw = \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w^{n+1}}\,dw\right]z^n$$

Similarly, for the integral around C_1

$$\frac{1}{w-z} = \frac{-1}{z}\cdot\frac{1}{1-\frac{w}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n.$$

Since for $w \in C_1$ we have $|w| = r_1 < |z|$, $\left|\frac{w}{z}\right| < 1$, & so the series converges (uniformly) in w.

Thus $$\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_1}\left[f(w)\cdot\left(-\frac{1}{z}\right)\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n\right]dw$$

$$= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{C_1}\left(-\frac{1}{z}\right)f(w)\left(\frac{w}{z}\right)^n dw$$

$$= -\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_1}f(w)w^n\,dw\right)\frac{1}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{C_1^+}f(w)w^n\,dw\right]\frac{1}{z^{n+1}}$$ where C_1^+ is C_1 with a positive orientation

$$= \sum_{k=-1}^{-\infty}\left[\frac{1}{2\pi i}\int_{C_1^+}\frac{f(w)}{w^{k+1}}\,dw\right]z^k$$

$\therefore$ By Cauchy's Thm. these integrals are all equal to $\frac{1}{2\pi i}\int_C \frac{f(w)}{w^{k+1}}\,dw$, so

$$f(w) = \sum_{k=-\infty}^{\infty}\left[\frac{1}{2\pi i}\int_C\frac{f(w)}{w^{k+1}}\,dw\right]z^k$$, as desired.

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