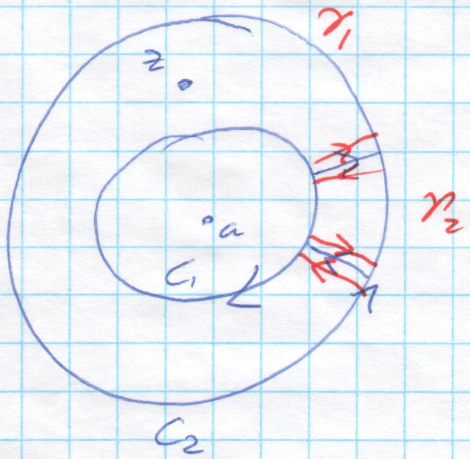
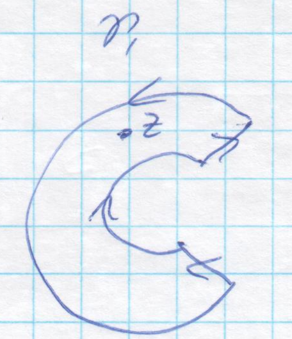
Lemma: Suppose $f(z)$ is holomorphic in the open annulus given by $T < |z-a| < U$ and C_1 & C_2 are circles with radii r_1 & r_2 resp. such that $T < r_1 < r_2 < U$ with $r_1 < |z-z_0| < r_2$, Where C_1 is negatively oriented & C_2 is positively oriented. Then

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw .$$





$$f(z) = \underbrace{\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw}_{\text{by Cauchy's Integral Formula \& Thm.}} + \underbrace{\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw}_{=0}$$

$$= \frac{1}{2\pi i}\int_{\gamma_1+\gamma_2} \frac{f(w)}{w-z}\,dw = \int_{C_1+C_2} \frac{f(w)}{w-z}\,dw$$

$+\cancel{\ell}+\cancel{\ell}$ cancel out

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw$$

//

Thm: Suppose $f(z)$ is a function that is holomorphic in the annulus $\{z \in \mathbb{C} \mid T < |z-a| < U\}$ for some $a \in \mathbb{C}$ and $T, U \in \mathbb{R}$ with $0 \leq T < U$.

Then $f(z)$ can be represented by a Laurent series centred at a, ie $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$, where the coefficients are given by

$$c_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{k+1}} \, dw ,$$

for any positively oriented circle with centre a and radius r, $T < r < U$.

proof:

To follow!