Laurent Series

... are what we get when we extend power series to allow for negative exponents.

Def'n: A **Laurent series** centred at $a \in \mathbb{C}$ is one of the form $\sum_{n=-\infty}^{\infty} c_n (z-a)^n$

$$= \cdots + c_{-2}(z-a)^{-2} + c_{-1}(z-a)^{-1} + c_0 + c_1(z-a) + \cdots$$

$$= \cdots + \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{(z-a)} + c_0 + c_1(z-a) + \cdots$$

Q.: For which z does a Laurent series $\sum_{n=-\infty}^{\infty} c_n (z-a)^n$ converge?

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n = \sum_{n=-\infty}^{-1} c_n (z-a)^n + \sum_{n=0}^{\infty} c_n (z-a)^n$$

We know that $\sum_{n=0}^{\infty} c_n (z-a)^n$ will converge absolutely for all z with $|z-a| < R$ for some radius of convergence R. What about $\sum_{n=-\infty}^{-1} c_n (z-a)^n$?

$$\sum_{n=-\infty}^{-1} c_n (z-a)^n = \sum_{k=1}^{\infty} \frac{c_{-k}}{(z-a)^k}$$

Claim: This will converge absolutely for z with $|z-a| > S$ for some $S \geq 0$.

Consequence: $\sum_{n=-\infty}^{\infty} c_n (z-a)^n$ cvgs abs. for z w. $S < |z-a| < R$

Thm: Suppose a function is holomorphic in the annulus $\{z \in \mathbb{C} \mid S < |z-a| < R\}$. Then $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$, where

$$c_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{k+1}}\, dw$$

for any circle C (positively oriented) centred at a and with radius r s.t. $S < r < R$.