Power Series & Holomorphic Functions

Last time:

Thm: If $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ has radius of cvgce. $R>0$, then $f(z)$ is holomorphic in the open disk $D = \{z \in \mathbb{C} \mid |z-a| < R\}$.

This time:

Thm: Suppose $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ has radius of cvgce $R>0$. Then $f'(z) = \sum_{n=\phi}^{\infty} n c_n (z-a)^{n-1}$ for any $z \in D = \{z \in \mathbb{C} \mid |z-a| < R\}$.

proof: Suppose $w \in D$. Then we can pick a $0 < r < R$ s.t. the circle $C = \{z \mid |z-w| < r\}$ is $\subsetneq D$. Since f is holomorphic in D, we apply Cauchy's Integral Formula for $f(w)$:

$$f'(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-w)^2}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{(z-w)^2}\cdot\left(\sum_{n=0}^{\infty} c_n(z-a)^n\right)dz$$

$$= \sum_{n=0}^{\infty} \frac{c_n}{2\pi i}\int_C \frac{(z-a)^n}{(z-w)^2}\,dz$$

$$= \sum_{n=0}^{\infty} c_n \cdot \frac{d}{dz}(z-a)^n \Big|_{z=w}$$

$$= \sum_{n=0}^{\infty} c_n\, n\,(z-a)^{n-1}\Big|_{z=w}$$

$$= \sum_{n=0}^{\infty} c_n n (w-a)^{n-1} \quad //$$

Prop: If $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ has radius of cgce $R>0$, then so does $f'(z) = \sum_{n=1}^{\infty} c_n n(z-a)^{n-1}$.

proof: The radius of cgce of $f'(z)$ is at least R since the previous argument shows it converges for every z with $|z-a|<R$.
It can't be $>R$, since $|nc_n| \geq |c_n|$ for all $n \geq 1$ - just use the Comparison Test... //

Expanding on the red-boxed bit a bit: ③

Suppose, by way of contradiction that the radius of cvgce. of $f'(z) = \sum_{n=1}^{\infty} c_n \cdot n \cdot (z-a)^{n-1}$ was some $S > R$.

Then the series would have to converge absolutely for any z with $S > |z-a| > R$,

ie we would have $\sum_{n=1}^{\infty} |c_n \cdot n \cdot (z-a)^{n-1}|$

$$= \sum_{n=1}^{\infty} |c_n| \cdot n \cdot |z-a|^{n-1}$$

converge. It would then follow that $\sum_{n=1}^{\infty} |c_n| \cdot n \cdot |z-a|^n = |z-a| \cdot \sum_{n=1}^{\infty} |c_n| \cdot n \cdot |z-a|^{n-1}$ also converges, and hence, by the Comparison Test, so does $\sum_{n=0}^{\infty} |c_n||z-a|^n$

because $|c_n| \cdot |z-a|^n \leq |c_n| \cdot n \cdot |z-a|^n$ for $n \geq 1$. But if $\sum_{n=0}^{\infty} |c_n||z-a|^n$ converged, so would $\sum_{n=0}^{\infty} c_n (z-a)^n$, which is impossible since $|z-a| > R$ and R is the radius of convergence of $\sum_{n=0}^{\infty} c_n (z-a)^n$.

Thus we must have $S \leq R$, as desired.