Convergence of Power Series

Defn: A <u>power series centred at $a \in \mathbb{C}$</u> is a series of the form $\sum_{n=0}^{\infty} c_n (z-a)^n$, where $c_i \in \mathbb{C}$ for $i \geq 1$.

Thm: Given a power series centred at $a \in \mathbb{C}$, say $\sum_{n=0}^{\infty} c_n (z-a)^n$, there exists an $R \geq 0$ (with $R = \infty$ possible) such that

$\sum_{n=0}^{\infty} c_n (z-a)^n$ converges absolutely for $|z-a| < R$ and diverges for $|z-a| > R$. Moreover, If $0 \leq r < R$, then the cvgce is uniform for all $|z-a| < r$.

[**Note:** for $|z-a| = R$, the series might converge or diverge for various such z ...]

Lemma: Suppose $\sum_{n=0}^{\infty} c_n (z-a)^n$ converges for some $z \in \mathbb{C}$, and $|w-a| < |z-a|$. Then $\sum_{n=0}^{\infty} c_n (w-a)^n$ converges absolutely.

pf: Since $\sum_{n=0}^{\infty} c_n (z-a)^n$ converges, $\lim_{n\to\infty} c_n (z-a)^n = 0$ by the Divergence Test, from which it follows that $\lim_{n\to\infty} |c_n (z-a)^n| = \lim_{n\to\infty} |c_n| \cdot |z-a|^n = \lim_{n\to\infty} |c_n| \cdot r^n = 0$ (where $r = |z-a|$). But then there must exist an $M \geq 0$ s.t. $|c_n| \cdot r^n \leq M$ for all n, so (as $|w-a| < |z-a| = r$), $\sum_{n=0}^{\infty} |c_n (w-a)|^n = \sum_{n=0}^{\infty} |c_n| \cdot r^n \left(\frac{|w-a|}{r}\right)^n$

$\leq M \sum_{n=0}^{\infty} \left(\frac{|w-a|}{r}\right)^n$, which is a cvgt. geometric series.

$\therefore \sum_{n=0}^{\infty} c_n (w-a)^n$ cvgs. abs. //

Pf: (of the Thm)

Let $R = \sup\{|z-a| \mid \sum_{n=0}^{\infty} c_n (z-a)^n \text{ cvgs.}\}$.

If $R = 0$, there is nothing to do...

Suppose $R > 0$ and $|z-a| < R$. By the defn of R, there is some z_1 with $|z-a| < |z_1 - a| \leq R$ s.t. $\sum_{n=0}^{\infty} c_n (z_1 - a)^n$ cvgs. By the Lemma, since $|z-a| < |z_1 - a|$ and $\sum_{n=0}^{\infty} c_n (z_1-a)^n$ cvgs, it follows that $\sum_{n=0}^{\infty} c_n (z-a)^n$ cvgs. absolutely, as required.

On the other hand, if $|z-a| > R$, then $\sum_{n=0}^{\infty} c_n (z-a)^n$ diverges by the defn of R.

[The "Moreover" part is left to the reader ... you can check the proof in the textbook, too.]

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