Convergence of Sequences & Series of Functions (1)

Pointwise vs Uniform Convergence

<u>Defn:</u> Suppose $G \subseteq \mathbb{C}$ and $\{f_n\}$ is a sequence of functions $f_n : G \to \mathbb{C}$. Then

(1) $\{f_n\}$ <u>converges pointwise</u> to some function $f : G \to \mathbb{C}$ if for all $z \in G$, $\lim_{n\to\infty} f_n(z) = f(z)$,

"$f_n \to f$ on G"

<u>ie</u> For ~~every~~ each $z \in G$ ~~and every~~ ε, ~~given~~ $\varepsilon > 0$, there is a N (for that ε) s.t. if $n \geq N$, then $|f_n(z) - f(z)| < \varepsilon$.

(2) $\{f_n\}$ <u>converges uniformly</u> to $f : G \to \mathbb{C}$

"$f_n \xrightarrow{unif} f$ on G"

if ~~for all~~ for each $\varepsilon > 0$, there is an N (for that ε) such that for all $z \in G$ if $n \geq N$, then $|f_n(z) - f(z)| < \varepsilon$.

That is, given an $\varepsilon > 0$ there is an N which works for that ε at every $z \in G$ simultaneously.

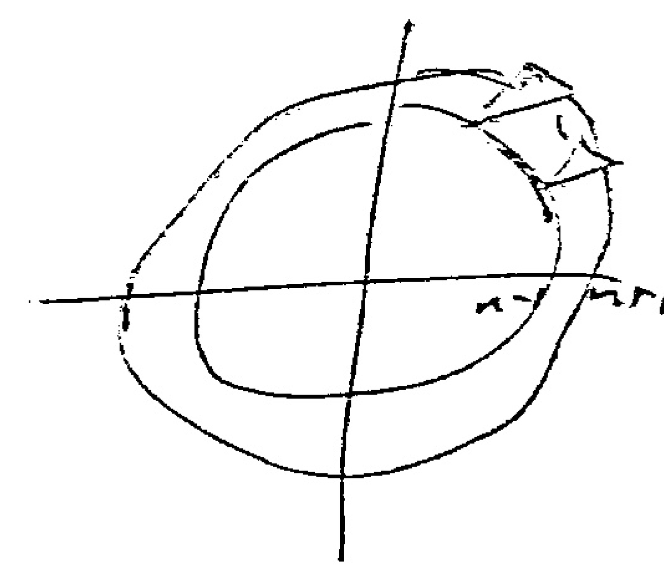
This is a much more powerful requirement than pointwise cvgce.

Why do we prefer uniform convergence?

It preserves more properties of functions...

$(n \geq 1)$

$$f_n(z) = \begin{cases} 0 & |z| \leq n-1 \\ \dfrac{(|z|-n+1)z}{|z|} & n-1 \leq |z| \leq n \\ \dfrac{n-|z|+1}{|z|} z & n \leq |z| \leq n+1 \\ 0 & |z| \geq n+1 \end{cases}$$



Note that $f_n(z)$ is cts & defined on $\mathbb{C}$

$$\lim_{n \to \infty} f_n(z) = 0 \quad \text{for all } z \in \mathbb{C}$$

so $f_n \to 0$ <u>pointwise</u>

But <u>not</u> uniformly: if $\varepsilon > 0$ and $\varepsilon < 1$, there are always $z \in \mathbb{C}$ for each n (e.g. $z = n + i0$) s.t.

$|f_n(z) - 0| \geq 1$ (so not $< \varepsilon$).

<u>Proposition:</u> Suppose $G \subseteq \mathbb{C}$ is a region and $\{f_n\}$ is a sequence of continuous functions $G \to \mathbb{C}$. If $f_n \xrightarrow{\text{unif}} f$ on G, then $f: G \to \mathbb{C}$ is also cts.

<u>proof:</u> Suppose $f_n \xrightarrow{\text{unif.}} f$ on G and each $f_n : G \to \mathbb{C}$ is continuous. We need to show that $f: G \to \mathbb{C}$ is continuous, too:

ie if $w \in G$ and $\varepsilon > 0$, there is a $\delta > 0$ s.t. if $|z-w| < \delta$, then $|f(z) - f(w)| < \varepsilon$.

Suppose we are given a $w \in G$ and an $\varepsilon > 0$. Then $\frac{\varepsilon}{3} > 0$ so there is an N s.t. $|f_n(z) - f(z)| < \frac{\varepsilon}{3}$ for all $n \geq N$. Pick an $n \geq N$. $f_n : G \to \mathbb{C}$ is continuous at $w \in G$ and $\frac{\varepsilon}{3} > 0$, so there is a $\delta > 0$ s.t. if $|z-w| < \delta$, then $|f_n(z) - f_n(w)| < \frac{\varepsilon}{3}$.

We claim this δ works for f & $\varepsilon > 0$ at w:
Suppose $|z-w| < \delta$, then:

$$
\begin{aligned}
|f(z) - f(w)| &\leq |f(z) - f_n(z) + f_n(z) - f(w)| \\
&\leq |f(z) - f_n(z)| + |f_n(z) - f(w)| \\
&< \frac{\varepsilon}{3} + |f_n(z) - f_n(w) + f_n(w) - f(w)| \\
&< \frac{\varepsilon}{3} + |f_n(z) - f_n(w)| + |f_n(w) - f(w)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

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Prop.: Suppose $\{f_n\}$ is a sequence of cts. fns. $G \to \mathbb{C}$ for some region, $f_n \xrightarrow{unif} f$ on G, and $\gamma \subset G$ is a piecewise smooth path. Then

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz.$$

proof: It's actually pretty easy. If γ is a single point, $\int_\gamma f_n(z)\,dz = 0 = \int_\gamma f(z)\,dz$, so the limit is trivial. If γ is not a single point, then length$(\gamma) > 0$. In this case, suppose $\varepsilon > 0$ is given. Since $f_n \xrightarrow{unif} f$ on G, there is an N s.t. for all $n \geq N$ and all $z \in G$

$$|f_n(z) - f(z)| < \frac{\varepsilon}{\text{length}(\gamma)}.$$

Then

$$\left|\int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz\right| = \left|\int_\gamma (f_n(z) - f(z))\,dz\right|$$

$$\leq \max\{|f_n(z) - f(z)| \mid z \in \{\gamma\}\} \cdot \text{length}(\gamma)$$

$$\leq \frac{\varepsilon}{\text{length}(\gamma)} \cdot \text{length}(\gamma) = \varepsilon,$$

so $\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz$. //

<u>Thm:</u> (Weierstrauss M-test)

Suppose $\{f_n\}$ is a sequence of functions $G \to \mathbb{C}$, $|f_n(z)| \leq M_n$ for all $z \in G$, and $\sum_{n=0}^{\infty} M_n$ cvgs. Then both $\sum_{n=0}^{\infty} |f_n(z)|$ and $\sum_{n=0}^{\infty} f_n(z)$ converge uniformly on G [ie the partial sums do].

[In this case, $\sum_{n=0}^{\infty} f_n(z)$ is said to <u>converge absolutely</u> <u>and uniformly</u> on G.]