A quick review of sequences and series
(for complex numbers)

Most of the definitions and many of the techniques and properties of sequences and series of complex numbers are the same as for the reals. (Not all, though.)

<u>Def'n:</u> If $\{a_n\}$ is a sequence of complex numbers and $L \in \mathbb{C}$, then $\lim_{n\to\infty} a_n = L$ means that for all $\varepsilon > 0$ ($\varepsilon \in \mathbb{R}$, here) there is an $N \geq 0$ ($N \in \mathbb{R}$, usually $N \in \mathbb{N}$) such that if $n \geq N$, then $|a_n - L| < \varepsilon$.

We will often write $a_n \to L$ and/or say that <u>a_n converges to L</u> if $\lim_{n\to\infty} a_n = L$. If there is no L s.t. $a_n \to L$, then $\{a_n\}$ is said to <u>diverge</u>.

Most of the basic properties of sequences work like they do in the reals

<u>eg</u> $\lim_{n\to\infty} (a_n \overset{+}{\underset{\div}{-}} b_n) = \left(\lim_{n\to\infty} a_n\right) \overset{+}{\underset{\div}{-}} \left(\lim_{n\to\infty} b_n\right)$, provided all the sequences converge (and you don't divide by 0... :-).

Similarly, if we want to add up a sequence $\{a_n\}$;

<u>Defn</u>: If $\{a_n\}$ is a <u>sequence</u> of complex numbers and $S \in \mathbb{C}$, then $\sum_{n=0}^{\infty} a_n = S$ means that $\lim_{n \to \infty} S_n = S$, where S_n is the partial sum up to a_n inclusive, i.e. $S_n = \sum_{i=0}^{n} a_i$.

We will often say that a series <u>converges</u> if there is such an $S \in \mathbb{C}$, and say it <u>diverges</u> otherwise. Most of the basic properties of series carry over from the real numbers, too, but a number of convergence test often used for real series cannot be applied directly to complex series, including the Comparison Tests and the Integral Test. In many cases we can apply these tests indirectly, by reducing our convergence problem to that for a real series. ~~The~~ One key tool here is the Cauchy criterion:

<u>Prop.</u>: (Cauchy criterion for convergence of <u>sequences</u>) Suppose $\{a_n\}$ is a sequence of complex numbers and for all $\varepsilon > 0$, there is an N such that for all $k \geq n \geq N$, $|a_n - a_k| < \varepsilon$. Then $\{a_n\}$ converges to some limit $L \in \mathbb{C}$.

Prop.: (Cauchy criterion for convergence of series)

Suppose $\sum_{n=0}^{\infty} a_n$ is a series such that for all $\varepsilon > 0$, there is an N such that for all $k \geq n \geq N$, $|S_k - S_n| = |a_{n+1} + a_{n+2} + \cdots + a_k| < \varepsilon$.

Then $\sum_{n=0}^{\infty} a_n$ converges to some sum $S \in \mathbb{C}$.

One quick consequence of this fact is the following result:

Thm: If $\sum_{n=0}^{\infty} |a_n|$ converges, so does $\sum_{n=0}^{\infty} a_n$.

This tells us that if a certain real series converges, so does a complex one, which lets us use things like the Comparison Test indirectly for complex series in many cases.