Harmonic Functions

Def'n: Suppose $G \subseteq \mathbb{C}$ (or, more generally, $G \subseteq \mathbb{R}^2$) is a region (ie a connected open set).

A real-valued function $u : G \to \mathbb{R}$ is said to be **harmonic** if it has continuous second partial derivatives in G and satisfies the Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \qquad \left[\text{or } u_{xx} = -u_{yy}, \text{ or } u_{xx} + u_{yy} = 1, \text{ or}\right] \ldots$$

Examples: $u(x,y) = x+y$, $u(x,y) = \cosh(x)\cos(y)$,

Note: Functions satisfying the Laplace equation (and its higher-dimensional versions) turn up in a lot of applications, including fluid flow, electrostatics, gravitation, and so on.

Harmonic functions have close connections with holomorphic/analytic functions.

Thm: Suppose $f(z) = u(z) + iv(z)$ is holomorphic in a region $G \subseteq \mathbb{C}$. Then both $u(z)$ and $v(z)$ are harmonic in G.

Note: The converse is not usually true. For example, $u(x+iy) = \ln(x^2+y^2)$ is harmonic (on $\mathbb{C} \setminus \{0\}$) but there is no harmonic ~~harmonic~~ $v(x+iy)$ such that $f(z) = u(z) + iv(z)$ is holomorphic (in $\mathbb{C} \setminus \{0\}$).

proof: Suppose $f(z) = u(z) + iv(z)$, for $u, v : G \to \mathbb{R}$, (2)
is holomorphic in G. One of the consequences of the extended version of the Cauchy Integral Formula, is that it follows that $f(z)$ is infinitely differentiable at every point $z \in G$. This implies that $\frac{\partial^2 f}{\partial x^2}$ & $\frac{\partial^2 f}{\partial y^2}$ (which are $\frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2}$ & $\frac{\partial^2 u}{\partial y^2} + i\frac{\partial^2 v}{\partial y^2}$) are both defined and continuous in G, as are the mixed partials $\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}$.

Since $f(z)$ is diff'ble in G, its component functions u & v satisfy the Cauchy-Riemann equations

$$u_x = v_y \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\& \quad u_y = -v_x, \quad \text{ie} \quad \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2}{\partial y \partial x} = 0$$

since the mixed second partials of v must be equal if ~~w~~ all the second partials of v are continuous.

A similar argument shows that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, too.

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A partial converse to this theorem does exist.

Prop: Suppose $u(x+iy)$ is harmonic in a simply-connected region G [ie a connected region with no holes]. Then there is a harmonic function $v(x+iy)$ in G such that $f(z) = u(z) + iv(z)$ is holomorphic in G.

Defn: The $v : G \to \mathbb{R}$ given by this theorem is called the harmonic conjugate of u.

proof: We will use $u(z)$, or rather it's partial derivatives, to help define $v(z)$.

Since $u(z)$ is harmonic it has continuous first and second partial derivatives.

First, let $g(z) = u_x(z) - iu_y(z) \underset{=}{\;} u_x(z) + i(-u_y(z))$. Then $g(z)$ is defined and continuous on the region G, since $u_x(z)$ & $u_y(z)$ are. Since the second partial derivatives of $u(z)$ are also defined and continuous in G, ~~we have~~ ~~u_{xx}~~ and u is harmonic, u_x & $-u_y$ satisfy the Cauchy-Riemann equations:

$$(u_x)_x = u_{xx} = -u_{yy} = (-u_y)_y$$

and $$(u_x)_y = u_{xy} = u_{yx} = -(-u_y)_x.$$

It follows that $g(z) = u_x(z) - iu_y(z)$ is holomorphic in G. Since G is simply ~~co~~ connected we can then define an antiderivative $h(z)$ for $g(z)$ via $\int_w^z g(z)dz$ for some $w \in G$.

Now suppose $h(z) = a(z) + ib(z)$.

Then $g(z) = h'(z) = a_x(z) + i b_x(z)$

$$= a_x(z) - i a_y(z)$$

$$g(z) = u_x(z) - i u_y(z)$$

$\Rightarrow \quad u(x,y) = a(x,y) + c(y)$ from $u_x = a_x$

& $\quad u(x,y) = a(x,y) + c(x)$ — " — $u_y = a_y$

$\Rightarrow \quad u(x,y) = a(x,y) + C$

Now let $f(z) = h(z) + C$

$$= a(z) + ib(z) + C$$

$$= (a(z) + C) + ib(z)$$

$$= u(z) + ib(z)$$

... and this is holomorphic in G because $h(z)$ is and has real part $u(z)$, as desired. ✓