Fundamental Theorem of Algebra

Suppose $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial in z with coefficients $a_n, a_{n-1}, \ldots, a_0 \in \mathbb{C}$ such that $a_n \neq 0$ & $n \geq 1$. Then $p(z)$ has a <u>root</u> (or <u>zero</u>) in $\mathbb{C}$, <u>ie</u> there is a $w \in \mathbb{C}$ s.t. $p(w) = 0$.

(Not the one actually delivered in the lecture...)

<u>proof</u>: Suppose, by way of contradiction, that some such a polynomial $p(z)$ has no root.

Then $\frac{1}{p(z)}$ is entire, and, so by Cauchy's Integral Formula we have

$$\frac{1}{p(0)} = \frac{1}{2\pi i} \int_{C_r} \frac{\frac{1}{p(z)}}{z}\, dz,$$

where C_r is the circle of radius r with centre 0, oriented positively.

This works for any and every $r > 0$. Then

$$\left|\frac{1}{p(0)}\right| = \frac{1}{2\pi} \left| \int_{C_r} \frac{1}{z p(z)}\, dz \right| \leq \frac{1}{2\pi} \left(\max\left\{ \left|\frac{1}{z p(z)}\right| \,\middle|\, z \in C_r \right\} \right) \cdot \underbrace{\text{length of } C_r}_{= 2\pi r}$$

Note that

$$z\,p(z) = a_n z^{n+1} + a_{n-1} z^{n} + \cdots + a_0 z$$

$$= a_n z^{n+1} \Big(\underbrace{1 + \frac{a_{n-1}}{a_n z} + \frac{a_{n-2}}{a_n z^2} + \cdots + \frac{a_0}{a_n z^{n+1}}}_{\to 1 + 0 \text{ as } |z| \to \infty} \Big),$$

so for some r large enough, if $z \in C_r$ (ie $|z| = r$), we have

$$\frac{|a_n| r^{n+1}}{2} = \frac{|a_n z^{n+1}|}{2} \leq |z p(z)| \leq |a_n z^{n+1}| \cdot \frac{3}{2} = \frac{3|a_n|}{2} r^{n+1}$$

Thus, for r large enough,

$$\frac{1}{|zp(z)|} \leq \frac{1}{\frac{|a_n| r^{n+1}}{2}} = \frac{2}{|a_n| r^{n+1}}$$

so $\max\left\{ \left|\frac{1}{zp(z)}\right| \,\middle|\, z \in C_r \right\} \leq \frac{2}{|a_n| r^{n+1}}$.

For all such r, we then have

$$0 \leq \left|\frac{1}{p(0)}\right| \leq \frac{1}{\cancel{2\pi}} \cdot \frac{2}{|a_n| r^{n+1}} \cdot \cancel{2\pi} r = \frac{2}{|a_n| r^n}$$

Thus, taking the limit as $r \to \infty$, we get

$$0 \leq \left|\frac{1}{p(0)}\right| \leq 0,$$

so $\left|\frac{1}{p(0)}\right| = 0$ ... which is a contradiction to ~~the idea that~~ how arithmetic in $\mathbb{C}$ works: dividing a (non-zero) number into 1 can't give you 0, since otherwise $0 \cdot p(0) = 1$ ...

$\therefore$ By contradiction, $\frac{1}{p(z)}$ cannot be entire, so $p(z)$ must have a zero in $\mathbb{C}$. //