Cauchy's Inequality

Suppose $f(z)$ is holomorphic in the region G and C is a positively oriented circle of radius r and with centre a such that $C \subsetneq G$. Let $M \in \mathbb{R}$ be a constant such that $|f(z)| \leq M$ for all $z \in C$.

Then $|f^{(n)}(a)| \leq \frac{M \cdot n!}{r^n}$ for all $n \geq 0$.

proof: By the extended version of Cauchy's Integral Formula, we have that

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz .$$

Recall the inequality

$$\left| \int_\gamma f(z) dz \right| \leq \max\{ |f(z)| \mid z \in \gamma \} \cdot \underbrace{\int_\gamma |\gamma'| dz}_{= \text{length of } \gamma}$$

Applying this inequality to the complex integral $\int_C \frac{f(z)}{(z-a)^{n+1}} dz$, we know that the length of C is $2\pi r$ and that

$\max\{ |f(z)/(z-a)^{n+1}| \mid z \in C \} \leq \frac{M}{r^{n+1}}$. Thus

$$|f^{(n)}(a)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz \right| = \frac{n!}{2\pi} \left| \int_C \frac{f(z)}{(z-a)^{n+1}} dz \right|$$

$$\leq \frac{n!}{\cancel{2\pi}} \cdot \frac{M}{\cancel{r^{n+1}}} \cdot \cancel{2\pi r} = \frac{n! \cdot M}{r^n}, \text{ as required.} //$$

Liouville's Theorem

A bounded entire function is constant.

[Recall that "entire" means that the function is holomorphic in all of $\mathbb{C}$.]

proof: We first show that $f'(w)=0$ for all $w \in \mathbb{C}$ if $f(z)$ is bounded and holomorphic in all of $\mathbb{C}$:

[Recall: $f(z)$ bounded on all of $\mathbb{C}$ means that for some $M \in \mathbb{R}$, $|f(z)| \leq M$ for all $z \in \mathbb{C}$.]

Let C_r be the circle of radius $r>0$ and centre $w \in \mathbb{C}$. By Cauchy's Inequality, (with $n=1$)

$$0 \leq |f'(w)| \leq \frac{1! \cdot M}{r^1} = \frac{M}{r}.$$

Since this is true for all $r>0$, $0 \leq |f'(w)| \leq \lim_{r\to\infty} \frac{M}{r} = 0$, so we must have $f'(w)=0$. Note that this works for all $w \in \mathbb{C}$ because $f(z)$ is entire.

Now suppose $a, w \in \mathbb{C}$ and $a \neq w$. We will show that $f(a)=f(w)$; this is sufficient to show that $f(z)$ must be constant. Let γ be the path $\gamma: [0,1] \to \mathbb{C}$ given by $\gamma(t) = a + t(w-a)$, so $\gamma(t)$ is differentiable and $\gamma(0)=a$ & $\gamma(1)=w$. Then

$$f(w) = f(a) + \int_\gamma f'(z)\,dz = f(a) + \int_\gamma 0\,dz = f(a) + 0 = f(a),$$

as required. //

Note that this is <u>very</u> unlike the situation of real-valued functions of one real variable, where there are plenty of bounded, everywhere differentiable functions, such as $f(x) = \frac{1}{1+x^2}$, $\cos(x)$, $\sin(x)$, & so on.

There are other differences, too. For example, there are plenty of non-constant polynomials $p(x)$ over the real numbers that do not have a root or zero, ie for which there is no value of $x \in \mathbb{R}$ s.t. $p(x) = 0$. $p(x) = 1+x^2$ is one such, for example, because $1+x^2 \geq 1 > 0$ for all $x \in \mathbb{R}$. However, every non-constant polynomial over the complex numbers does have a root:

<u>The Fundamental Theorem of Algebra</u>

Suppose $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree $n \geq 1$ over the complex numbers [ie with all the coefficients $a_n, a_{n-1}, \ldots, a_0 \in \mathbb{C}$]. Then there is at least one $w \in \mathbb{C}$ s.t. $p(w) = 0$, ie $p(z)$ has a root in $\mathbb{C}$.