Thm: Suppose $f(z)$ is holomorphic in the region $G \subseteq \mathbb{C}$, and γ is a positively oriented, simple, closed, & piecewise smooth path in G which is contractible (in G) to a single point in G.

Then, for w inside γ, we have

$$f'(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^2}\, dz,$$

and, in general for $n \geq 0$,

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{n+1}}\, dz.$$

Note: The case $n=0$ is just Cauchy's Integral Formula,

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\, dz.$$

(Recall that $0! = 1$ ...)

(& $0! = 1$, too)

Lemma: Suppose $f(z)$, G, γ, & w, are as in the statement of the theorem. Then, for $n \in \mathbb{Z}$,

$$\int_\gamma (z-w)^n \, dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = 1 \end{cases}$$

proof: Recall from Assignment # 5, question 1, that if γ is the unit circle in $\mathbb{C}$, then

$$\int_\gamma z^n \, dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1. \end{cases}$$

... take it from there.

[Use a homotopy to move γ to a positively oriented circle C_r with radius r centred at w. Then use the substitution

$$s = \frac{z-w}{r}$$

to ~~modif~~ get the problem into a form where one can use the result from A#5.]

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Assuming all the derivatives exist and $f(z)$ is equal to its Taylor series at w:

pf of thm:
$$f(z) = \sum_{n=0} \frac{f^{(n)}(w)}{n!}(z-w)^n$$

$$= f(w) + f'(w)(z-w) + \frac{f''(w)}{2!}(z-w)^2 + \cdots$$

$$\int_\gamma \frac{f(z)}{(z-w)^2}\,dz = \int_\gamma \frac{f(w)}{(z-w)^2}\,dz + \int_\gamma \frac{f'(w)(z-w)}{(z-w)^2}\,dz$$

$$+ \int_\gamma \frac{f''(w)}{2!}\frac{(z-w)^2}{(z-w)^2} + \cdots$$

$$\int_\gamma z^n\,dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n=-1 \end{cases}$$

$$= \int_\gamma \frac{f(w)}{(z-w)^2} + \int \frac{f'(w)}{z-w}\,dz$$

$$+ \int_\gamma \frac{f''(w)}{2!}\,dz + \int \frac{f'''(w)}{3!}(z-w)\,dz + \cdots$$

by A#5 9.1

$$= 0 + f'(w)2\pi i + 0 + 0 + \cdots$$

$$\Rightarrow f'(w) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-w)^2}\,dz$$

Similarly, $$f^{(n)}(w) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-w)^{n+1}}\,dz$$

$n \geq 0$

~~Consequences~~

Consequences of Cauchy's Integral ~~f~~ormula

1. (Morera's Thm.) If $f(z)$ is cts in a region ~~G~~ G & $\int_\gamma f(z)\,dz = 0$ for every simple closed ~~c~~ curve in G, then $f(z)$ is holomorphic in G.

2. Cauchy's Inequality

If $f(z)$ is ~~analy~~ holomorphic in G & C is a positively oriented circle of radius r in G centred at a,

Then $$|f^{(n)}(a)| \leq \frac{M \cdot n!}{r^n} \quad \text{for } n = 0, 1, 2, \ldots$$

where M is a constant such that $|f(z)| \leq M$ on the circle C.

3. Liouville's Thm

Suppose that $f(z)$ is holomorphic on $\mathbb{C}$ & bounded on $\mathbb{C}$. Then $f(z)$ is constant.

(Show $f'(z) = 0$.)