Cauchy's Integral Formula

2022-02-17

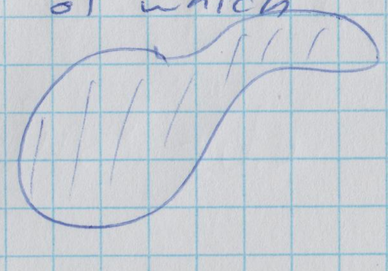
or, the weirdness of complex integration gets worse!

Recap: Cauchy's Thm: If f is holomorphic in a region $G \subseteq \mathbb{C}$, and η & γ are piecewise ~~continuous~~ smooth closed curves which are homotopic in G ($\eta \sim_G \gamma$), then $\int_\gamma f = \int_\eta f$.

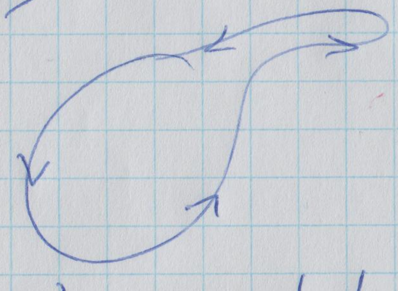
One consequence is that if γ is homotopic to a point in G (ie γ is G-contractible, $\gamma \sim_G 0$), then $\int_\gamma f = 0$.

Recall: A curve γ in $\mathbb{C}$ is simple if it doesn't cross itself.

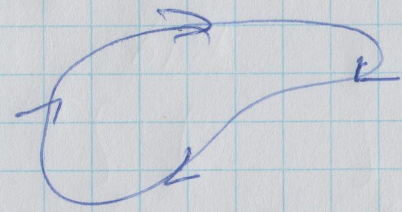
Jordan Curve Thm: A simple closed path in $\mathbb{C}$ divides $\mathbb{C}$ into two open sets, each of which is connected, and of which one is $\underbrace{\text{bounded}}_{\text{interior of the curve}}$ and one is $\underbrace{\text{unbounded}}_{\text{exterior of the curve.}}$



Defn. A simple closed path in $\mathbb{C}$ is *positively oriented* if it's parametrized so that when you traverse the path using this parametrization, the interior of the curve is always on the left.



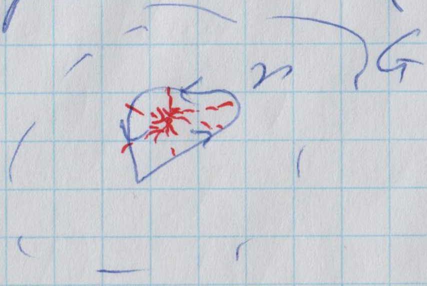
+'ve orientation



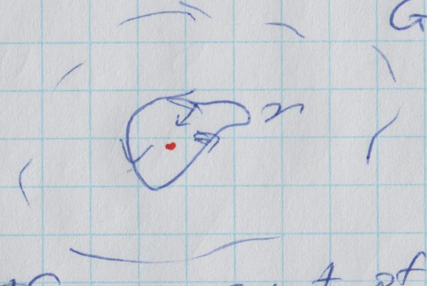
-'ve orientation

Prop: If γ is a simple closed piecewise smooth path in a region $G \subseteq \mathbb{C}$, then G contains all of the interior of γ if and only if γ is a contractible to a point in G ($\gamma \sim_G 0$).

proof: $\boxed{\Rightarrow}$



$\boxed{\Leftarrow}$



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If some point of the interior is not in G, then any homotopy contracting γ to a point must pass through the missing point...

Cauchy's Integral Formula

Suppose $f(z)$ is holomorphic in a region $G \subseteq \mathbb{C}$,
γ is a piecewise smooth simple closed curve with positive orientation in G,
and w is a point in the interior of γ (& hence $w \in G$ too).

Then $$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\, dw .$$

Why???

proof 1: See the text...

proof 2: **Lemma:** $\int_C \frac{1}{z-w}\, dz = 2\pi i$

p.f. Via change of variables, $s = z - w$, & bit of homotopy to make C be the unit circle,

$$\int_C \frac{1}{z-w}\, dz = \int_{e^{i\theta},\ 0\le\theta\le 2\pi} \frac{1}{z}\, dz = 2\pi i \ \ldots \quad //$$

Now define a function $g : G \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z) - f(w)}{z - w} & z \neq w \\[2ex] f'(w) & z = w \end{cases}$$

Since $f'(w) = \lim\limits_{z \to w} \dfrac{f(z) - f(w)}{z - w}$ by def'n of $f'(w)$,

$g(z)$ is continuous at w and holomorphic on $G \setminus \{w\}$.

[This is good enough to be able to shrink γ to w in G and evaluate $\int_\gamma g$ by evaluating ~~g at~~ $\int_w g$.]

Then $0 = \int_\gamma g(z)\,dz = \int_\gamma \dfrac{f(z) - f(w)}{z - w}\,dw$

$\parallel$ w is not on γ

$\int_w g(z)\,dz$

constant?

$$= \int_\gamma \frac{f(z)}{z - w}\,dz - \int \frac{f(w)}{z - w}\,dz$$

$$0 = \int \frac{f(z)}{z-w}\,dz - f(w) \cdot \int \frac{1}{z-w}\,dz$$

& by the Lemma...

$$= \int \frac{f(z)}{z-w}\,dz - f(w) \cdot 2\pi i$$

$$\Rightarrow \quad \int \frac{f(z)}{z-w}\,dz = 2\pi i\, f(w)$$

$$\Rightarrow \quad f(w) = \frac{1}{2\pi i} \int \frac{f(z)}{z-w}\,dz. \qquad //$$

What does this formula do for us?

Wait for it!