Cauchy Thm. and its consequences

Recall:

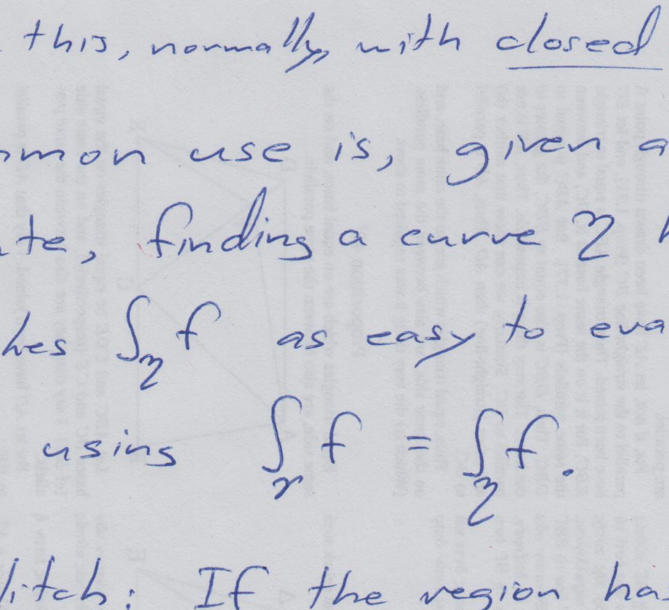
<u>Cauchy's Thm.</u>: (The somewhat careful version...)

Suppose $G \subseteq \mathbb{C}$ is a region (<u>ie</u> a path-connected open set), $f: G \to \mathbb{C}$ is holomorphic in G (<u>ie</u> $f'(z)$ is defined at all $z \in G$), and γ & η are piecewise smooth curves which are homotopic in G (<u>ie</u> we can continuously deform γ into η without going outside G). [Notation: $\gamma \sim_G \eta$] Then $\int_\gamma f = \int_\eta f$.

(We will use this, normally, with <u>closed</u> curves.)

One common use is, given an integral $\int_\gamma f$ to compute, finding a curve η homotopic to γ which makes $\int_\eta f$ as easy to evaluate as possible and then using $\int_\gamma f = \int_\eta f$.

<u>Possible glitch</u>: If the region has holes or missing points, then closed curves may not be homotopic in the region:



None of these are homotopic <u>in</u> G.

However, if there are no missing points or larger holes in the way of the homotopy in G, a closed curve will be homotopic to a point. $O \to o \to \cdot$

In this case a curve γ in G is said to be <u>G-contractible</u> (Textbook notation: $\gamma \sim_G 0$), and it follows from Cauchy's Thm. that

$$\int_\gamma f = 0 \quad [\text{Why?}].$$

In particular, if $f(z)$ is an <u>entire</u> function (ie holomorphic on $\mathbb{C}$), then $\int_\gamma f = 0$ on any [piecewise smooth] curve γ in $\mathbb{C}$.

<u>eg</u> Suppose γ is the unit circle centred at the origin in $\mathbb{C}$, positively oriented [ie counterclockwise].

Then
$$\int_\gamma \frac{1}{z^2-4} = \int_\gamma \frac{}{z-2}\,dz + \int \frac{}{z+2}\,dz$$

$$= 0 + 0 \quad [\text{because?}]$$

On the other hand, if γ is the circle of radius 1 in $\mathbb{C}$ centred at $2+0i$, then

$$= \int_\gamma \frac{}{z-2}\,dz + \int \frac{}{z+2}\,dz$$

$$= \_\_\,2\pi i + 0$$