Complex Integration II

Defn: If F is holomorphic in some region G and $F'(z) = f(z)$ for all $z \in G$, then $F(z)$ is an antiderivative of f on G.

Eg

Thm: Suppose $\gamma \subseteq G \subseteq \mathbb{C}$ is a piecewise smooth curve in G with parametrization $\gamma(t)$, $t \in [a,b]$.

If f is continuous on G and F is an (any!) antiderivative of f on G, then

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$
$$= F(\gamma(b)) - F(\gamma(a)).$$

Corollary: If $G \subseteq \mathbb{C}$ is open, $\gamma \subseteq G$ is a piecewise smooth closed curve, f is continuous on G and has an antiderivative on G, then $\int_\gamma f = 0$.

Book uses example of $\frac{1}{z} : \mathbb{C}\setminus\{0\} \to \mathbb{C}$.

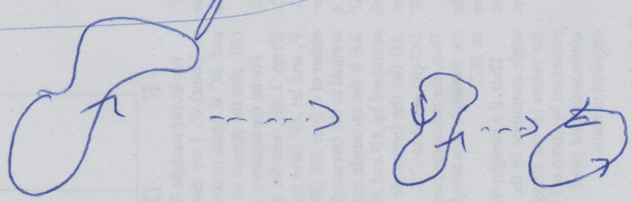
Since $\int_{\partial\mathbb{D}} \frac{1}{z}\,dz = 2\pi i \neq 0$, $\frac{1}{z}$ does not have an antiderivative on $\mathbb{C}\setminus\{0\}$. It does have one on ~~es~~ $\mathbb{C}\setminus\{x \mid x \in \mathbb{R} \;\&\; x \leq 0\}$.

unit circle

<u>Thm:</u> Suppose $G \subseteq \mathbb{C}$ is a region & $z_0 \in G$, and let $f: G \to \mathbb{C}$ be a cts. fn. s.t $\int_\gamma f = 0$ for any piecewise smooth closed curve in G. Then $F: G \to \mathbb{C}$ defined by $F(z) = \int_{\gamma_z} f$, where γ_z is any piecewise smooth path from z_0 to z) is an antiderivative for f on G.

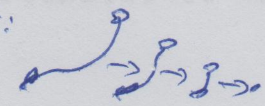
Cauchy's Theorem

First, two paths (in some region $G \subseteq \mathbb{C}$) are homotopic if you can continuously deform & move one path [without going outside G!] to become the other. eg



[illustrate with rubber band]

We'll only need this for closed paths.
[non-closed paths that are finite are all homotopic to a point:



...so they're usually boring in this context.]

Officially: If γ_0 & γ_1 are closed paths in G with parametrizations $\gamma_0 : [0,1] \to \mathbb{C}$ & $\gamma_1 : [0,1] \to \mathbb{C}$, then γ_0 & γ_1 are G-homotopic (or homotopic in G) if there is a cts. fn. $h : \underbrace{[0,1] \times [0,1]}_{= \{(t,s) \mid t,s \in [0,1]\}} \to G$

s.t. $\forall s,t \in [0,1]$,
$$h(t,0) = \gamma_0(t)$$
$$h(t,1) = \gamma_1(t)$$
$$\& \; h(0,s) = h(1,s)$$
to ensure each intermediate curve is closed

Notation: $\gamma_0 \sim_G \gamma_1$ if γ_0, γ_1 are G-homotopic.

Cauchy Thm is then the followings: ②

Suppose $G \subseteq \mathbb{C}$ is a region (ie a path-connected open set),

$f: G \to \mathbb{C}$ is holomorphic in G,

and γ_0, γ_1 are piecewise smooth curves which are ~~holomorphic~~ homotopic in G (ie $\gamma_0 \sim_G \gamma_1$).

Then $\int_{\gamma_0} f = \int_{\gamma_1} f$.

<u>Use:</u> This suggests the strategy of, when computing $\int_\gamma f$, of finding the curve η ~~holo~~ homotomic to γ that makes $\int_\eta f$ as easy to evaluate as possible and then using $\int_\gamma f = \int_\eta f$ ...