Complex Integration

This will stick to the integration along curves $\mathbb{C}$ inherits from $\mathbb{R}^2$, so the basic def'n is:

Def'n: Suppose γ is a (piecewise) smooth curve parametrized by $\gamma : [a,b] \to \mathbb{C}$ and f is a complex function $f(z) = u(z) + iv(z)$ $[u, v : \mathbb{C} \to \mathbb{R}]$ which is continuous on γ. Then the integral of f on γ is

$$\int_\gamma f = \int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$

$$= \int_a^b u(\gamma(t)) \cdot \gamma'(t)\,dt + i\int_a^b v(\gamma(t)) \cdot \gamma'(t)\,dt.$$

(example)

Prop.: Suppose γ & η are different parametrizations of the same (piecewise) smooth curve

i.e. $\gamma : [a,b] \to \mathbb{C}$ & $\eta : [c,d] \to \mathbb{C}$ are both piecewise smooth, $\gamma(a) = \eta(c)$ & $\gamma(b) = \eta(d)$ and $\{\gamma(t) \mid t \in [a,b]\} = \{\eta(t) \mid t \in [c,d]\}$

Then $\int_\gamma f = \int_\eta f$.

(example)

Def'n: The <u>length</u> of the smooth (or piecewise smooth) curve γ is

$$\int_\gamma |\gamma'| = \int_a^b |\gamma'(t)|\,dt \quad (\text{if } \gamma : [a,b] \to \mathbb{C}).$$

(example)

Properties of the complex integral:

1) $\displaystyle\int_\gamma (af + bg) = a\int_\gamma f + b\int_\gamma g \qquad a, b \in \mathbb{C}$

2) If γ is γ_1 followed by γ_2 [note this requires the end of γ_1 to match with the beginning of γ_2], then

$$\int_\gamma f = \int_{\gamma_1\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f,$$

3) If $-\gamma$ is γ run in reverse
[es $\gamma : [a,b] \to \mathbb{C} \Rightarrow -\gamma : [a,b] \to \mathbb{C}$ is given by $(-\gamma)(t) = \gamma(a+b-t)$],
then $\displaystyle\int_{-\gamma} f = -\int_\gamma f.$

4) $\displaystyle\left|\int_\gamma f\right| \leq \max\{|f(z)| \mid z \in \gamma\} \cdot \underbrace{\int_\gamma |\gamma'|}_{\text{is length of } \gamma}$

(examples)

e.g. Let C be the unit circle oriented positively, parametrized, say, by

$$r(t) = (\cos(t), \sin(t)) \quad \text{for } t \in [0, 2\pi].$$

Let $P(x,y) = x^2 y$ and $Q(x,y) = y^2 + 3x$ for $(x,y) \in \mathbb{R}^2$.

(Think of them as the components of a vector valued function $\vec{F}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$.)

$$\int_C P\,dx + Q\,dy = \int_C [P(x,y)\vec{i} + Q(x,y)\vec{j}] \cdot d\vec{r}$$

where $\vec{r}$ is the position vector of the point on the curve.

$$= \int_0^{2\pi} [P(x,y)\vec{i} + Q(x,y)\vec{j}] \cdot \overrightarrow{r'(t)}\, dt$$

$$= \int_0^{2\pi} \left(P(x,y) \cdot \cos'(t) + Q(x,y) \sin'(t)\right) dt$$

$$= \int_0^{2\pi} \left[\cos^2(t)\sin(t) \cdot (-\sin(t)) + (\sin^2(t) + 3\cos(t))\cos(t)\right] dt$$

$$= \int_0^{2\pi} \left[-\cos^2(t)\sin^2(t) + \sin^2(t)\cos(t) + 3\cos^2(t)\right] dt$$

etc

Corollary: $\int_C P\,dx + Q\,dy$ ~~for some curve C~~ is independent of the particular curve C [so it depends only on the starting and endpoints]

iff $\vec{F}(x,y) = (P(x,y), Q(x,y))$ is conservative as a force

exactly when

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

[for all (x,y) in some open region containing C]