Integrating complex functions?

Quick review of bits of vector calculus, etc. first.

We inherit a lot from integrating functions $f: \mathbb{R}^2 \to \mathbb{R}$. Recall that there are two main forms of integration in that case:

1) Double integrals over some region in $\mathbb{R}^2$

eg
$$\iint_{[0,1]\times[0,1]} xy \, dA = \int_0^1 \int_0^1 xy \, dx \, dy$$
$$= \int_0^1 \frac{x^2}{2} \cdot y \Big|_0^1 dy = \int_0^1 \left(\frac{1^2}{2}y - \frac{0^2}{2}y\right) dy$$
$$= \int_0^1 \frac{y}{2} dy = \frac{y^2}{4}\Big|_0^1 = \frac{1^2}{4} - \frac{0^2}{4} = \frac{1}{4}$$

2) Line integrals over some parametrized curve.

eg Suppose $\gamma(t) = (2t+1, t^2)$ for $t \in [1,2]$ is a parametrized curve in $\mathbb{R}^2$, and $f(x,y) = xy$.

Then
$$\int_\gamma f = \int_1^2 (f\circ\gamma)(t)\, dt = \int_1^2 f(2t+1, t^2)\, dt$$
$$= \int_1^2 (2t+1)(t^2)\, dt = \int_1^2 (2t^3 + t^2)\, dt$$
$$= \left(2 \cdot \frac{t^4}{4} + \frac{t^3}{3}\right)\Big|_1^2 = \left(\frac{2^4}{2} + \frac{2^3}{3}\right) - \left(\frac{1^4}{2} + \frac{1^3}{3}\right)$$
$$= \frac{16}{2} + \frac{8}{3} - \frac{1}{2} - \frac{1}{3} = \frac{15}{2} + \frac{7}{3} = \frac{45+14}{6} = \frac{59}{6}.$$

There are a number of theorems about each type of integral of $\mathbb{R}^2$ (eg Fubini's Thm for $\iint$), and one, in particular, that relates them:

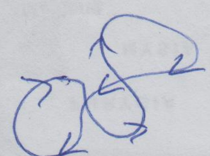
<u>Green's Thm:</u> [as stated in Stewart's <u>Calculus</u>]

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open set containing D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA .$$

=

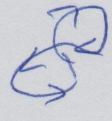
<u>Terms:</u> "closed curve"



it comes back to itself no matter where you start on it

"piecewise smooth"
it can be decomposed into finitely many pieces that each have some differentiable parametrization

"simple"
it doesn't cross over itself

eg  is simple,  is not

"positively oriented"
when you go around the curve with the interior [the <u>bounded</u> region it's the border of] on the left-hand side. (Counter-clockwise, more or less)