3770H

Logarithms

<u>Def'n.</u> Let $\mathrm{Arg}(z) = \theta$ for $z = re^{i\theta} \neq 0$ and the ~~unique~~ $\theta \in (-\pi, \pi]$.
(This is the <u>principal argument</u> of z; any other θ is just an "argument.")

The <u>principal</u> (branch of the) <u>logarithm</u> (function) is

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z).$$

Is this an inverse of e^z?

Suppose $e^z = e^{x+iy} = e^x e^{iy}$ for $y \in (-\pi, \pi]$.

Then $\mathrm{Log}(e^z) = \ln|e^x| + i\,\mathrm{Arg}(e^{iy})$

$= x + iy.$

Note that if $z = x+iy$ with $y \notin (-\pi, \pi]$,

then $\mathrm{Log}(e^z) \neq z$.

More generally any function $f: G \to \mathbb{C}$
[for G a "region" = connected open set] s.t.
$e^{f(z)} = z$ is a <u>branch of the logarithm</u> (on G).

[Region = connected open set]

Note that $\text{Log}(zw)$ (& logarithms in general) need not equal $\text{Log}(z) + \text{Log}(w)$ [unlike the real case]

Sometimes it does, eg $\text{Log}(-i \cdot i) = 0 = \text{Log}(-i) + \text{Log}(i)$ [work out]

[or $\text{Log}(i(i+1)) = \text{Log}(i) + \text{Log}(i+1)$]

& sometimes it doesn't, eg $\text{Log}(i(i-1)) \neq \text{Log}(i) + \text{Log}(i-1)$.

Note that Log, in particular, is <u>not</u> continuous for $z = x + i0$ where $x \in \mathbb{R}$ & $x \leq 0$.

<u>Fact:</u> If log is a branch of the logarithm, then it is differentiable [except at 0] and

$$\frac{d}{dz} \log(z) = \frac{1}{z}.$$

$$\mathrm{Log}(-i\cdot i) = \mathrm{Log}(-(-1)) = \mathrm{Log}(1)$$

$$= \ln(1) + i\,\mathrm{Arg}(e^{i\cdot 0})$$

$$= \cancel{\ln 1}\ 0 + i\cdot 0 = 0$$

$$\mathrm{Log}(-i) = \mathrm{Log}(1\cdot e^{i\cdot(-\pi/2)})$$

$$= \ln(1) + i\,\mathrm{Arg}(e^{i\cdot(-\pi/2)})$$

$$= 0 \cancel{+} -i\frac{\pi}{2}$$

$$\mathrm{Log}(i) = \mathrm{Log}(1\cdot e^{i\pi/2})$$

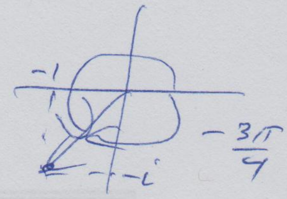
$$= \ln(1) + i\,\mathrm{Arg}(e^{i\pi/2})$$

$$= 0 + i\frac{\pi}{2}$$

So $\mathrm{Log}(-i\cdot i) = 0 = \cancel{0} -i\frac{\pi}{2} + i\frac{\pi}{2}$

$$= \mathrm{Log}(-i) + \mathrm{Log}(i)$$

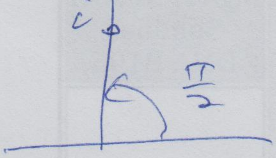
$$\operatorname{Log}(i(i-1)) = \operatorname{Log}(-1-i)$$



$$= \operatorname{Log}\left(\sqrt{2}\cdot e^{i(-3\pi/4)}\right)$$

$$= \ln(\sqrt{2}) + i\operatorname{Arg}\left(e^{i(-3\pi/4)}\right)$$

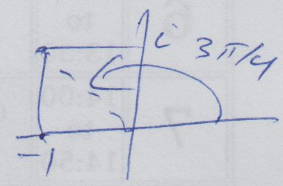
$$= \frac{1}{2}\ln(2) + i\left(-\frac{3\pi}{4}\right)$$



$$\operatorname{Log}(i) = \operatorname{Log}\left(1e^{i\pi/2}\right)$$

$$= \ln(1) + i\operatorname{Arg}\left(e^{i\pi/2}\right)$$

$$= 0 + i\frac{\pi}{2} = i\frac{\pi}{2}$$



$$\operatorname{Log}(i-1) = \operatorname{Log}(-1+i)$$

$$= \operatorname{Log}\left(\sqrt{2}e^{i3\pi/4}\right)$$

$$= \ln(\sqrt{2}) + i\operatorname{Arg}\left(e^{i3\pi/4}\right)$$

$$= \frac{1}{2}\ln(2) + i\frac{3\pi}{4}$$

$$\operatorname{Log}(i) + \operatorname{Log}(i-1) = i\frac{\pi}{2} + \frac{1}{2}\ln(2) + i\frac{3\pi}{4}$$

$$= \frac{1}{2}\ln(2) + i\frac{5\pi}{4}$$

$$\neq \frac{1}{2}\ln(2) + i\left(-\frac{3\pi}{4}\right)$$

$$= \operatorname{Log}(i(i-1))$$

[Though note that $e^{i5\pi/4} = e^{-i3\pi/4}$.]

We can extend the notion of exponentiation to all complex numbers via ($a, b \in \mathbb{C}$)

$$a^b = e^{b \operatorname{Log}(a)}$$

[this is the principal branch of a^b - other log fns. give other branches]

Can we do logarithms to other bases?

Sure!

$$\log_a(z) = b \iff a^b = e^{b \operatorname{Log}(a)} = z$$

Is there such a b? If so, is it unique?