3770H Exponential & trig & hyperbolic fns.

$$e^z = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

$z = x+iy$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

Note that, unlike e^x for $x \in \mathbb{R}$, e^z is <u>not</u> 1-1 since $e^{x+iy} = e^{x+i(y+2k\pi)}$ for all $k \in \mathbb{Z}$.

[This means complications down the road for defining logarithms, not unlike those we have in defining inverse functions for the real-valued trig functions, as you have to pick the branch you want to invert.]

Otherwise, the usual properties of the exponential fn. ~~is~~ $\frac{d}{dz}e^z = e^z$ [& is entire (is diff'ble and defined everywhere)] still work. One addition to them is $|e^z| = |Re(e^z)|$ $\left(= e^x \text{ if } z = x+iy\right)$

Recall that for real x we had the hyperbolic functions $\cosh(x) = \dfrac{e^{x}+e^{-x}}{2}$

& $\sinh(x) = \dfrac{e^{x}-e^{-x}}{2}$

etc

In the complex numbers, we can define $\sin(x)$, $\cos(x)$, etc in a similar way from e^{z}:

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = -i\cdot\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$$

etc Note that $\cos(z)$ & $\sin(z)$ are entire because e^{z} is.

* [Check: do these defns work for real inputs?]

Note that all the usual identities apply...

Hyperbolic fns have the same def'n from the exponential fn. as in the reals: $\cosh(z) = \dfrac{e^{z}+e^{-z}}{2}$, $\sinh(z) = \dfrac{e^{z}-e^{-z}}{2}$ etc and are also entire & satisfy the same identities as in the reals.

Note that $\sin(z) = -i\sinh(iz)$ & $\cos(z) = \cosh(iz)$

Note that both the trig & hyperbolic functions are <u>not</u> bounded on $\mathbb{C}$.