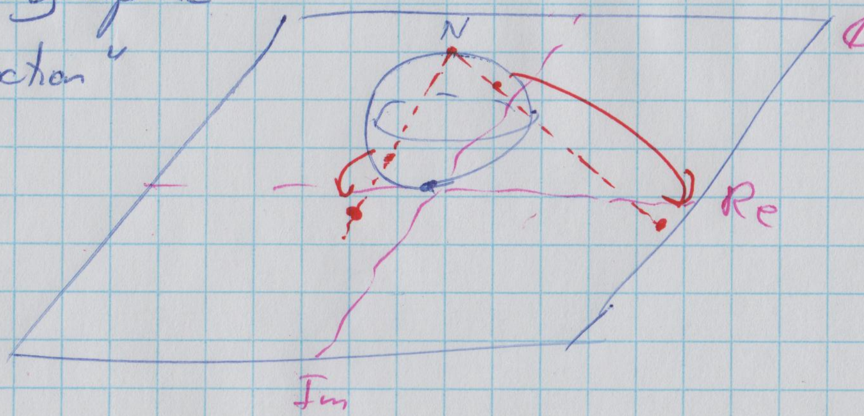
The extended complex plane,

also known as the Riemann sphere.

"stereographic projection"



Run the projection in reverse to map $\mathbb{C}$ onto the sphere, for the North Pole (N) we have to add a new symbol, ∞.

Defn: The extended complex plane or Riemann sphere is the set $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, equipped with the following extensions to the algebraic operations on $\mathbb{C}$:

i. $\infty + a = a + \infty = \infty$ for any & every $a \in \mathbb{C}$.

ii. $\infty \cdot a = a \cdot \infty = \infty$ — " — with $a \neq 0$.

iii. $\infty \cdot \infty = \infty$

iv. $\frac{a}{\infty} = 0$ for every $a \in \mathbb{C}$

v. $\frac{a}{0} = \infty$ for ever $a \in \mathbb{C}$ with $a \neq 0$.

Note that $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty + \infty$, & $0 \cdot \infty$ are all undefined; basically because it would mess up the properties of limits if you tried.

<u>Def'n:</u> Suppose f is a complex function. Then

1) $\lim_{z \to z_0} f(z) = \infty$ if for every $M > 0$ $(M \in \mathbb{R})$ there is a $\delta > 0$ $(\delta \in \mathbb{R})$ such that if $0 < |z - z_0| < \delta$, then $|f(z)| > M$.

2) $\lim_{z \to \infty} f(z) = a$ (for some $a \in \mathbb{C}$) if for every $\varepsilon > 0$, there is an $N > 0$ $(N \in \mathbb{R})$ such that if $|z| > N$, then $|f(z) - a| < \varepsilon$.

3) $\lim_{z \to \infty} f(z) = \infty$ if for every $M > 0$ $(M \in \mathbb{R})$ there is an $N > 0$ $(N \in \mathbb{R})$ such that if $|z| > N$, then $|f(z)| > M$.

<u>Examples:</u> (i) $\lim_{z \to 1+i} \frac{1}{z-(1+i)} = \infty$ because if $M > 0$,

let $\delta = \frac{1}{M}$ (so $\delta > 0$). Then if

$0 < |z-(1+i)| < \delta = \frac{1}{M}$, we have

$$\left| \frac{1}{z-(1+i)} \right| = \frac{1}{|z-(1+i)|} > \frac{1}{\delta} = \frac{1}{1/M} = M. \checkmark$$

(ii) $\lim_{z \to \infty} \frac{z-3}{iz+2} = \frac{1}{i} = -i$ [Note that $i \cdot \frac{1}{i} = 1$ & $i(-i) = -(-1) = 1$.]

Suppose $\varepsilon > 0$. We'll find the N we need by reverse-engineering:

$$\left| \frac{z-3}{iz+2} - (-i) \right| < \varepsilon \iff \left| \frac{z-3}{iz+2} + i \right| < \varepsilon$$

but $\left| \frac{z-3}{iz+2} + i \right| = \left| \frac{z-3+i(iz+2)}{iz+2} \right| = \left| \frac{\cancel{z}-3+\overset{-1}{i^2}\cancel{z}+2i}{iz+2} \right|$

$$= \left| \frac{-3+2i}{iz+2} \right| = \frac{|-3+2i|}{|iz+2|} = \frac{\sqrt{(-3)^2+2^2}}{|iz+2|} = \frac{\sqrt{13}}{|iz+2|}$$

so $\left|\dfrac{z-3}{iz+2} - (-i)\right| < \varepsilon \iff \dfrac{\sqrt{13}}{|iz+2|} < \varepsilon$

$\iff \sqrt{13} < \varepsilon \cdot |iz+2|$

$\iff |iz+2| > \dfrac{\sqrt{13}}{\varepsilon}$

since $|iz+2| > |iz| - |2|$ by rearranging the triangle inequality,

$\Longleftarrow |iz| - |2| > \dfrac{\sqrt{13}}{\varepsilon}$

$\iff |iz| > \dfrac{\sqrt{13}}{\varepsilon} + |2| = \dfrac{\sqrt{13}}{\varepsilon} + 2$

$= |i|\,|z|$

$\iff |z| > \dfrac{\sqrt{13}}{\varepsilon} + 2$

This means that if we make $N \geq \dfrac{\sqrt{13}}{\varepsilon} + 2$,

then $|z| > N$ guarantees that

$\left|\dfrac{z-3}{iz+2} - (-i)\right| < \varepsilon$, so the limit works.

<u>Note:</u> In general, $\lim\limits_{z \to \infty} \dfrac{az+b}{cz+d} = \dfrac{a}{c}$, ~~(as long as $c \neq 0$~~

[general proof in the text in §3.2]

(iii) $\lim_{z\to\infty} z^2 = \infty$ Suppose $M>0$. We find the needed $N>0$ by reverse-engineering:

$$|z^2| > M \iff |z|^2 > M$$
$$\iff |z| > \sqrt{M},$$

so $N \geq \sqrt{M}$ works here.

How does the Riemann sphere $\hat{\mathbb{C}}$ play with Möbius transformations?

Recall that a fractional ~~l~~ linear transformation
$$f(z) = \frac{az+b}{cz+d} \quad (\text{for some } a,b,c,d \in \mathbb{C})$$
is a Möbius transformation if $ad-bc \neq 0$.

A Möbius transformation like this can be extended to a nice 1-1 onto function $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as follows:

$$f(z) = \begin{cases} \dfrac{az+b}{cz+d} & , \text{if } z \in \mathbb{C} \text{ and } z \neq -\dfrac{d}{c}. \\ \infty & , \text{if } z = -\dfrac{d}{c} \\ \dfrac{a}{c} & , \text{if } z = \infty \end{cases}$$

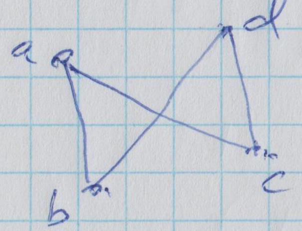
[Note that $-\infty = \infty$ in this scheme, because there is only one point in $\hat{\mathbb{C}}$ infinitely far from 0.]

provided $c \neq 0$

If $c = 0$, then $f(z) = \begin{cases} \dfrac{az+b}{d} & , \text{if } z \in \mathbb{C} \\ \infty & , \text{if } z = \infty \end{cases}$.

<u>Def'n:</u> If $a, b, c, d \in \hat{\mathbb{C}}$ and b, c, d are different from each other then their <u>cross-ratio</u> is

$$[a,b,c,d] = \frac{(a-b)(c-d)}{(a-c)(b-d)}. \quad (\text{If } a = c, \text{ this } = \infty.)$$



(If any one of a, b, c, d is ∞ then terms involving are cancelled out. eg $d = \infty$ then $[a,b,c,d] = \dfrac{a-b}{a-c}$.)

<u>Proposition:</u> [3.12 in the text] Suppose $f(z) = [z, b, c, d]$ for some fixed $b, c, d \in \hat{\mathbb{C}}$. Then $f(z)$ is a Möbius transformation with $f(b) = 0$, $f(c) = 1$, & $f(d) = \infty$. Moreover any Möbius transformation $g: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $g(b) = 0$, $g(c) = 1$, & $g(d) = \infty$ is <u>equal</u> to $f(z)$.

<u>pf:</u> text. //

<u>Corollary:</u> Every ~~linear~~ Möbius transformation can be written as the function defined by cross-product as above.

<u>Corollary:</u> Suppose $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ & $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ are two sets of three distinct elements of $\hat{\mathbb{C}}$ each. Then there is an unique Möbius transformation $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ s.t.

$f(z_1) = w_1$, $f(z_2) = w_2$, & $f(z_3) = w_3$.