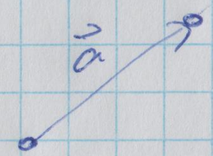
Transformations of the Complex Plane

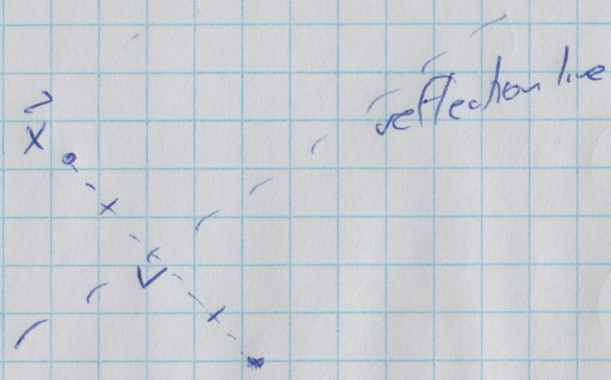
$\mathbb{R}^2$ has a fair number of well-known transformations:

translations:



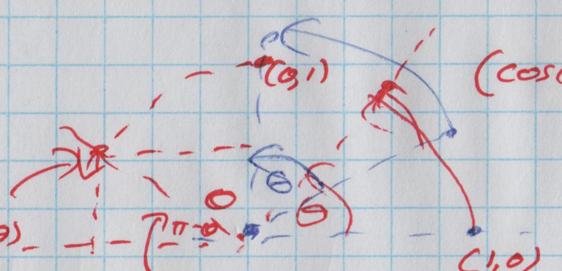
$\vec{x} \longmapsto \vec{a} + \vec{x}$

reflections:



eg reflection in the line $y=0$ (i.e. the x-axis) takes (u,v) to $(u,-v)$.

rotations:

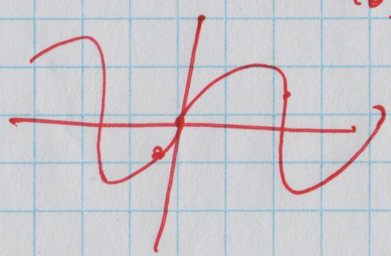


$\sin(\pi-\theta)$
$= -\sin(\theta-\pi)$
$= --\sin(\theta)$
$= \sin(\theta)$

$\cos(\pi-\theta)$
$= -\cos(\theta)$

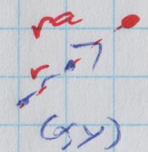
eg rotations about $(0,0)$ are given by multiplication by a matrix

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$



These are the "rigid motions" that don't change relative distances between points or angles between lines

<u>dilatation:</u> a dilotation with factor a about (x,y)



es dilatation about $(0,0)$ are easy

$(x,y) \longmapsto (ax, ay))$

We can do all of this in $\mathbb{C}$, since $\mathbb{C}$ is basically $\mathbb{R}^2$ with some additional structure (ie multiplication among the points).

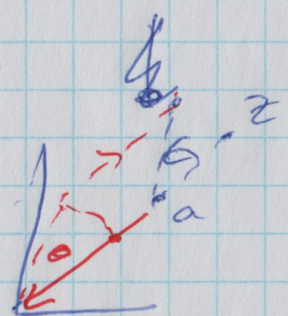
<u>Translations:</u> $z \longmapsto a+z$ (for some fixed $a \in \mathbb{C}$)

(Diff'ble!)

<u>Rotations:</u> By an angle of θ about O: $z \mapsto e^{i\theta} z$

[It's multiplication by a constant with absolute value 1.]

[Diff'ble!]



By an angle of θ about $a \in \mathbb{C}$:

$$z \mapsto z-a \longmapsto e^{i\theta}(z-a) \mapsto e^{i\theta}(z-a)+a$$

ie $z \mapsto e^{i\theta} z + a(1-e^{i\theta})$ [Diff'ble!]

Dilatation: A dilatation of factor $r \in \mathbb{R}$ about O:

$$z \mapsto rz$$

A dilatation of factor $r \in \mathbb{R}$ about a:

$$z \mapsto z - a \mapsto r(z-a) \mapsto r(z-a)+a$$

i.e. $z \mapsto rz + (1-r)\vec{a}$

A dilatation of factor $w_0 \in \mathbb{C}$ about O:

$$z \mapsto w_0 z = re^{i\theta}z \qquad \text{" } re^{i\theta} \text{ for some } r, \theta \in \mathbb{R}$$

... i.e. it's a rotation of angle θ about O coupled with a dilatation of factor $r = |w_0|$ about O.

General dilatation in $\mathbb{C}$

A dilatation of factor $w_0 \in \mathbb{C}$ about $\vec{a}$:

$$z \mapsto z - a \mapsto w_0(z-a) \mapsto w_0(z-a)+a$$

i.e. $z \mapsto w_0 z + (1-w_0)\vec{a}$

[Also diff'ble.]

Reflections: Not usually diff'ble:

eg $z \longmapsto \bar{z}$ reflection in the x-axis

$ \overset{\shortparallel}{x+iy} \quad \overset{\shortparallel}{x-iy}$ ... and it's not diff'ble

Inversions: $z \longmapsto \frac{1}{z}$

- an inversion about 0

Diff'ble for $z \neq 0$.

this sends a complex number a distance of r from 0 to a point a distance of $\frac{1}{r}$ from 0 ie it sends the complex plane outside $|z|=1$ to the complex plane inside $|z|=1$ & vice versa

Inversion about $a \in \mathbb{C}$:

$$z \longmapsto z-a \longrightarrow \frac{1}{z-a} \longrightarrow \frac{1}{z-a}+a$$

Diff'ble for $z \neq a$.

Def'n: A fractional linear transformation of $\mathbb{C}$ is a function $f(z) = \frac{az+b}{cz+d}$, where $a,b,c,d \in \mathbb{C}$ and ~~ct~~ it not the case that $c=0$ & $d=0$.

- A common generalization of our transformations (other than reflections).

Prop: Any fractional linear transformation ~~to~~ can be ~~es~~ written as a composition of rotations, translations, & dilatations.

proof: If $c=0$ (so $d \neq 0$), $f(z) = \frac{a}{d} z + \frac{b}{d}$

= general dilation [multiply z by $\frac{a}{d}$] followed by a translation [add $\frac{b}{d}$]

[Diff'ble]

If $c \neq 0$,

$$f(z) = \frac{az+b}{cz+d} = \frac{az + \frac{da}{c} - \frac{da}{c} + b}{cz+d}$$

$$= \frac{az + \frac{da}{c}}{cz+d} + \frac{b - \frac{da}{c}}{cz+d}$$

$$= \frac{\frac{1}{c}(caz+da)}{cz+d} + \left(b - \frac{da}{c}\right)\cdot\frac{1}{cz+d}$$

$$= \frac{a}{c}\cdot\frac{\cancel{cz+d}}{\cancel{cz+d}} + \left(\frac{bc-ad}{c}\right)\cdot\frac{1}{cz+d}$$

$$= \frac{a}{c} + \frac{bc-ad}{c^2}\cdot\frac{1}{z+\frac{d}{c}}$$

$$z \mapsto \underbrace{z + \frac{d}{c}}_{\text{translation}} \underbrace{\mapsto \frac{1}{z+\frac{d}{c}}}_{\text{inversion}} \underbrace{\mapsto \frac{bc-ad}{c^2}\cdot\frac{1}{z+\frac{d}{c}}}_{\text{general dilation}} \underbrace{\mapsto \frac{bc-ad}{c^2}\cdot\frac{1}{z+\frac{d}{c}} + \frac{a}{c}}_{\text{translation}}$$

[Diff'ble] [Invertible too]

If $ad-bc \neq 0$, then $f(z) = \frac{az+b}{cz+d}$ is called a Möbius transformation. Thm 3.4 in the text: If $f(z)$ is a Möbius transformation, then it takes circles & lines to circles & lines [possibly interchanging some circles with some lines]