Some properties of complex derivatives

or, one reason this is not entirely like dealing with real functions

Recall: 1) The derivative of $f(z)$ at z_0 is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{if it exists.}$$

- this requires $f(z)$ be defined at z_0 and for all z in some disk centred at z_0.

2) $f(z)$ is holomorphic at z_0 if it is differentiable at all z in some disk centred at z_0.

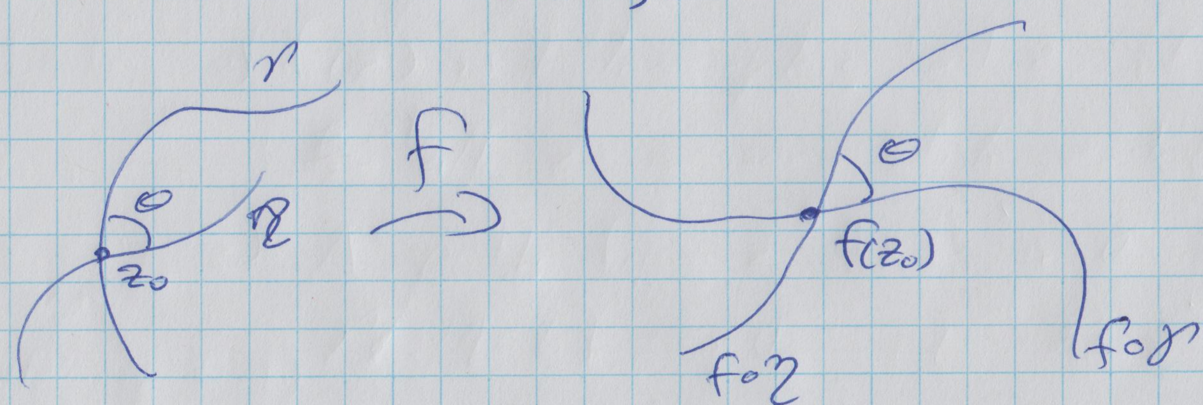
3) A path from z_0 to w_0 in $\mathbb{C}$ is a ~~smooth~~ ~~differentiable~~ continuous function $\gamma : [a,b] \to \mathbb{C}$ s.t. $\gamma(a) = z_0$ & $\gamma(b) = w_0$

4) A smooth path is a path with a differentiable parametrization (i.e. γ is differentiable on (a,b)).

Proposition: (2.11, in the text)

[γ & η make an angle of θ at some z_0 if their derivatives at this point make an angle of θ.]

Suppose $f(z)$ is holomorphic at z_0 with $f(z_0) \neq 0$. Assume γ and η are two smooth paths that pass through z_0, making an angle of θ with each other at this point. Then $f \circ \gamma$ and $f \circ \eta$ are also smooth paths and these make an angle of θ with each other at $f(z_0)$.



ie holomorphic functions preserve angles, that is, they are <u>conformal</u>.

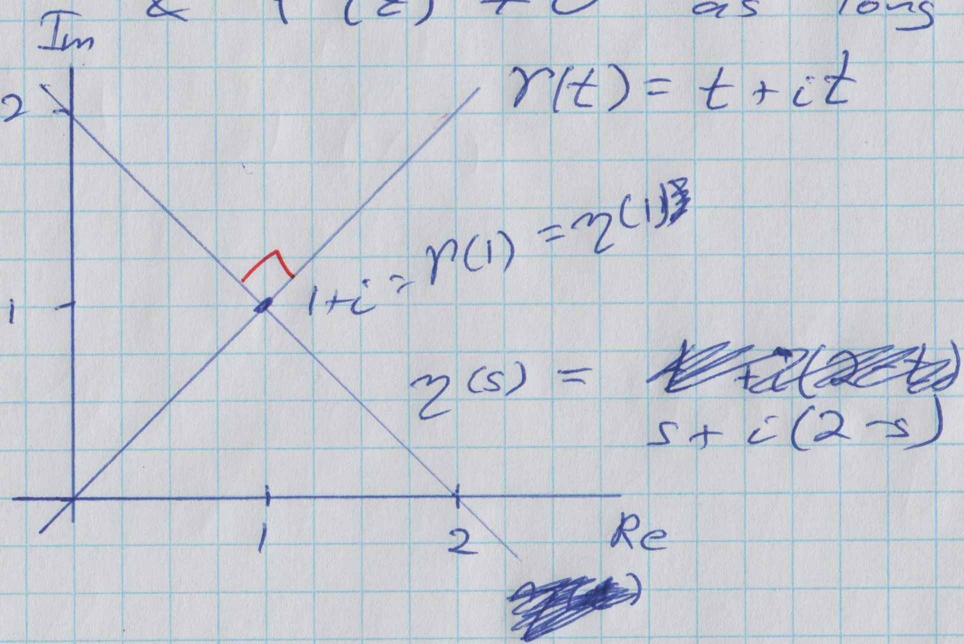
<u>Note:</u> Not every conformal function is differentiable

<u>eg</u> $f(z) = \bar{z}$ (ie $f(x+iy) = x - iy$) [reflection in the real axis] [so it preserves angles] is not differentiable.

Example: Consider $f(z) = z^2$

$$f(x+iy) = (x^2-y^2) + i\,2xy,$$

which is differentiable at all points in $\mathbb{C}$ & $f'(z) \neq 0$ as long as $z \neq 0$.



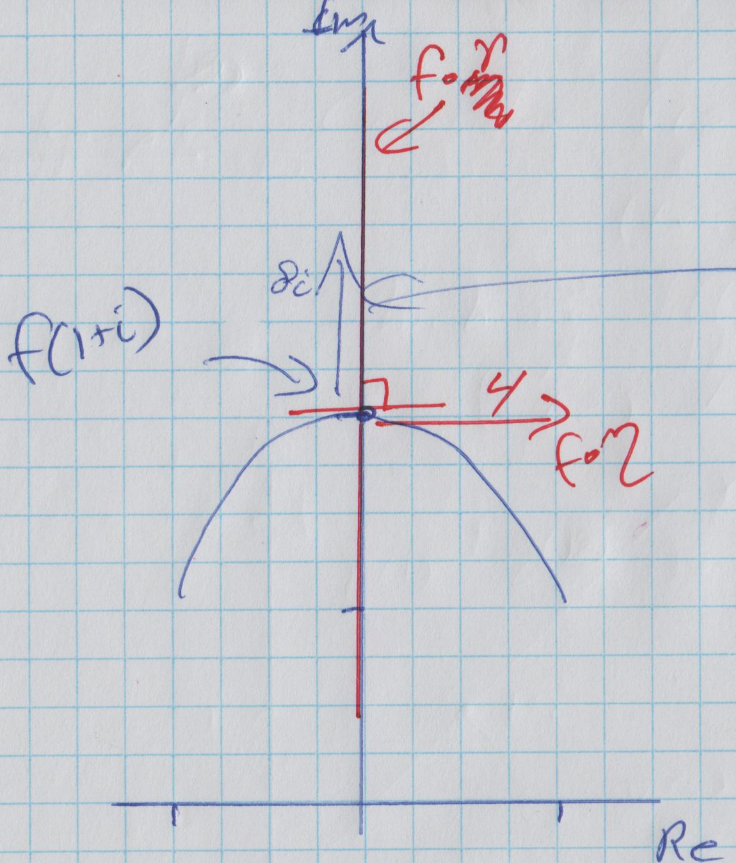
γ & η are differentiable (smooth) curves that cross at $z_0 = 1+i$, at right angles.

$$(f\circ\gamma)(t) = f(\gamma(t)) = f(t+it) = \cancel{t^2 - t^2} + 2it^2 = i(2t^2)$$

$$(f\circ\eta)(s) = f(\eta(s)) = f(s+i(2-s)) = s^2-(2-s)^2 + i2s(2-s)$$

$$= s^2 - (4-4s+s^2) + i2(2s-s^2) = (4s-4) + i(4s-2s^2)$$

$$x = 4s-4$$
$$y = 4s-2s^2$$



$$(f\circ\gamma)'(t) = \frac{d}{dt}(4t^2 i)$$
$$= i\cdot 8t$$
$$(f\circ\gamma)'(1) = 8i$$

$$(f\circ\eta)'(s) = \frac{d}{ds}\left((4s-4)+i(4s-2s^2)\right)$$
$$= 4 + i(4-4s)$$
$$(f\circ\eta)'(1) = 4 - i(4-4\cdot 1) = 4$$

So these are still perpendicular to each other at $f(1+i) = 2i$.

<u>proof</u> (of the Proposition): We can assume that the curves γ & η use the parameters t & s respectively and that $\gamma(c) = \eta(d) = z_0$. [ie the curves cross at z_0]
Then $\gamma'(c)$ & $\eta'(d)$ are the tangent vectors to the curves at z_0, and θ is the angle between these vectors.

Note that $f\circ\gamma$ and $f\circ\eta$ are going to be defined & differentiable for t near c & s near d since $f\circ\gamma$ & $f\circ\eta$ are defined near c & d, and are differentiable near c & d since they are compositions of differentiable functions.

The tangent vectors to $f\circ\gamma$ and $f\circ\eta$ at $f(z_0)$ are given by

Chain Rule!

$$\frac{d}{dt}f(\gamma(t))\Big|_{t=c} = f'(\gamma(c))\cdot\gamma'(c) = f'(z_0)\cdot\gamma'(c)$$

& $$\frac{d}{ds}f(\eta(s))\Big|_{s=d} = f'(\eta(d))\cdot\eta'(d) = f'(z_0)\cdot\eta'(d)$$

~~Suppose~~ $f'(z_0)\neq 0$ and suppose $f'(z_0)=re^{i\alpha}$.

What does multiplying $\gamma'(c)$ & $\eta'(d)$ by $f'(z_0)=re^{i\alpha}$ do? 1) Since $f'(z_0)\neq 0$, $r\neq 0$ (so $r>0$), ~~multiplying~~ multiplying $\gamma'(c)$ & $\eta'(d)$ by $f'(z_0)$ scales them both by r, which doesn't change angles...

2) The $e^{i\alpha}$ component rotates ~~everything~~, including $\gamma'(c)$ & $\eta'(d)$ by an angle of θ, which doesn't the relative angle between the two vectors.

Thus the angle between the tangent vectors to $f\circ\gamma$ & $f\circ\eta$ at $f(z_0)$ is the same as the angle between γ & η at z_0. //

Cauchy-Riemann equation(s)
(f is diff'ble at $z_0 = x_0 + iy_0$)

(1) $\frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$

(2) $f(z) = u(z_0) + iv(z_0)$

then (1) amounts

$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0)$

& $\frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)$

$\frac{\partial f}{\partial x}(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)$

$\frac{\partial f}{\partial y}(z_0) = \frac{\partial u}{\partial y}(z_0) + i\frac{\partial v}{\partial y}(z_0)$

& plug into (1)
to get (2)
or vice versa

These give us a slightly indirect way of checking if a function is diff'ble.

<u>Thm:</u> Suppose $f(z)$ is diff'ble at z_0.
Then the partial derivatives (of the components) of $f(z)$ ~~are~~ exist and satisfy the Cauchy-Riemann equation.

<u>Thm.:</u> Suppose $f(z)$ is a complex fn. s.t. $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist (in some open disk about z_0) and are continuous at z_0. Then if they satisfy the Cauchy-Riemann equation as well, the $f(z)$ is differentiable at z_0.

<u>proof:</u> see the textbook ... //