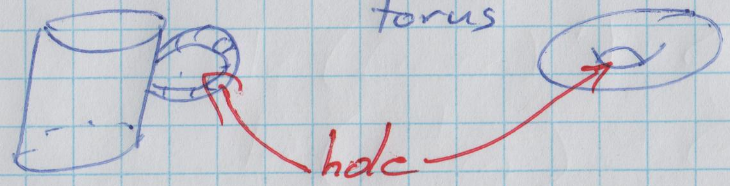
(A little) Topology in the Complex Plane

Topology is the study of properties that are invariant under continuous transformations.

eg coffee cup torus



are topologically equivalent since one could be continously deformed (no cutting, tearing, or glueing) (but bending & squeezing are ok) into the other

Having a hole is an example of a topological property

We will focus on the "real" basics: aiming for continuity for functions in the complex plane.

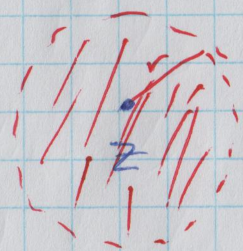
The "topology" of the complex plane is derived from the ~~distance~~ function: $d(z, w) = |z - w|$ (the Euclidean metric on $\mathbb{C}$)

metric

"Basic open sets": Open disks, (interiors of disks)

For $z \in \mathbb{C}$ & $r > 0$ in $\mathbb{R}$: $B_r(z)$ = open disk around z with radius r

$= \{ w \in \mathbb{C} \mid |z - w| < r \}$



Note this does not include any point on the boundary

We could rewrite the definition of limits in these terms: $\lim_{z \to z_0} f(z) = w_0$

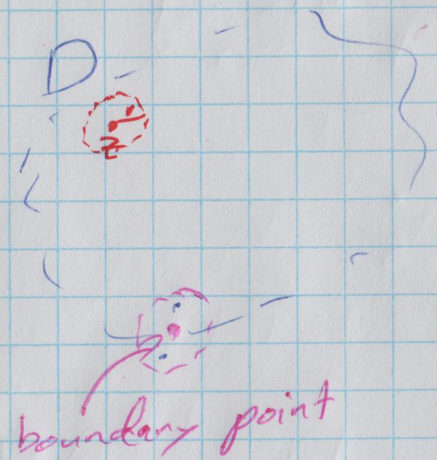
means that for every $\varepsilon > 0$ $(\varepsilon \in \mathbb{R})$

there is a $\delta > 0$ $(\delta \in \mathbb{R})$

such that if $z \in B_\delta(z_0)$, then $f(z) \in B_\varepsilon(w_0)$.

In general, a set $D \subseteq \mathbb{C}$ is <u>open</u> if it's a union of "basic open sets"

<u>ie</u> If $z_0 \in D$, there is some $r \in \mathbb{R}$, $r > 0$, such that $B_r(z_0) = \{w \mid |w - z_0| < r\} \subseteq D$.



If we say that (for any set $D \subseteq \mathbb{C}$) that z is a <u>boundary point</u> of D whenever $B_r(z)$ always contains points both in and outside of D for every r, then an open set does not contain <u>any</u> of its boundary points

A set is <u>closed</u> if it's complement is open.

<u>ie</u> D is closed if $\mathbb{C} \setminus D$ is open

Note that a closed set must include all of its boundary points.

We can define these notions in terms of sequences as well:

Def'n: A sequence $\{z_n\}$ has limit w_0 (in $\mathbb{C}$) if for all real $\varepsilon > 0$, there is an integer N s.t. for all $n \geq N$, $|z_n - w_0| < \varepsilon$

["$\lim_{n \to \infty} z_n = w_0$"] [i.e. $z_n \in B_\varepsilon(w_0)$].

Then a set D is closed iff whenever $\{z_n\}$ is a sequence of points of D and $\lim_{n \to \infty} z_n = w_0$, ~~then~~ we also have that $w_0 \in D$.

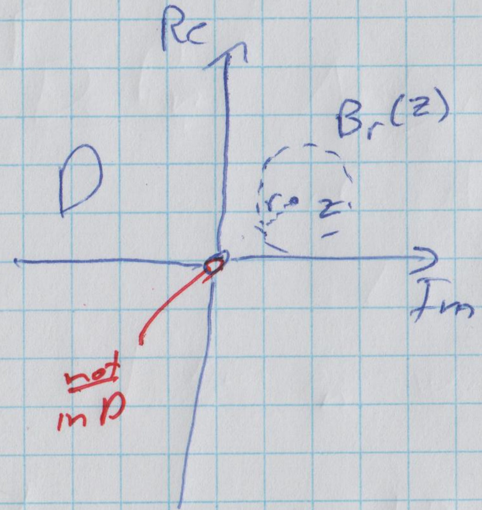
& then a set D is open iff $\mathbb{C} \setminus D$ is closed.

The closure of a set $D \subseteq \mathbb{C}$, is

$$\overline{D} = D \cup \{\text{all the boundary points of } D\}$$
$$= \bigcap \{E \mid D \subseteq E \text{ and } E \text{ is closed}\}$$

eg $f(z) = \frac{1}{z}$ is defined for all $z \neq 0$.

ie its domain is $D = \{ z \in \mathbb{C} \mid z \neq 0 \}$



This set D is open:

Given $z \in D$, $z \neq 0$, so let

$r = \frac{|z-0|}{2}$, and then

$B_r(z) \subseteq D$.

$\bar{D} = \mathbb{C}$ since 0 is a boundary point: [the only one!]

For any real $\varepsilon > 0$, $B_\varepsilon(0)$ contains non-zero complex numbers (hence elements of D)

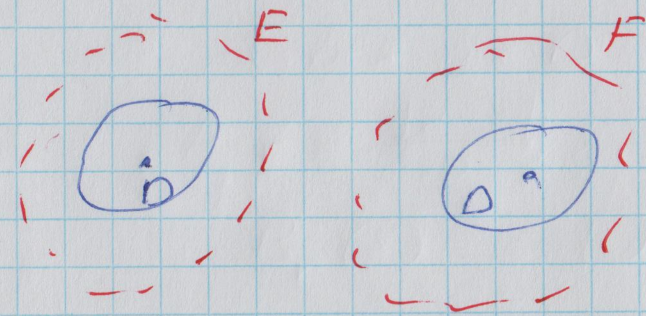
as well as a point, namely 0 itself, not in D.

The $\bar{D} = D \cup \{\text{all boundary points}\} = D \cup \{0\} = \mathbb{C}$.

⑥

A set $D \subseteq \mathbb{C}$ is <u>connected</u> if you cannot find two open sets E & F of $\mathbb{C}$ such that $D \cap E \neq \emptyset$ & $D \cap F \neq \emptyset$ but $E \cap F = \emptyset$.

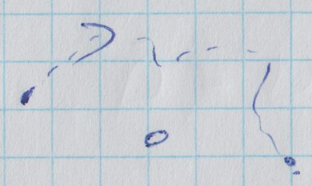
Cheap example:



This D is "disconnected".

Informally, you can't get from one part of D to ~~another~~ the other without passing outside of D, and there is real separation involved.

<u>Example:</u> $D = \{z \in \mathbb{C} \mid z \neq 0\}$ is connected



Paths: A *path* from z_0 to w_0 in $\mathbb{C}$ is (the graph of) a continuous function

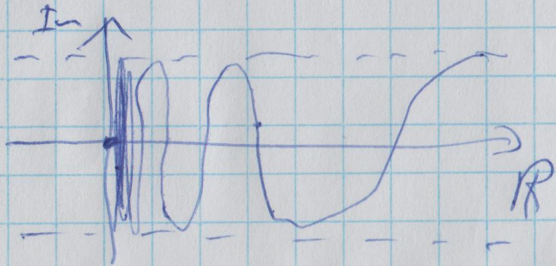
$([a,b] \subseteq \mathbb{R})$ $\quad f: [a,b] \to \mathbb{C}$ s.t. $f(a) = z_0$ & $f(b) = w_0$.

Def'n: A set $D \subseteq \mathbb{C}$ is *path-connected* if for any two points z_0 & $w_0 \in D$, there is a path from z_0 to w_0 which lies entirely in D. [ie if $f: [a,b] \to \mathbb{C}$ gives the path, then $\{f(x) \mid x \in [a,b]\} \subseteq D$.]

Fact: Path-connected $\Rightarrow$ connected

$$\not\Leftarrow$$

eg the "Warsaw sine curve" $\{x + i\sin(\frac{1}{x}) \mid x \in \mathbb{R}, x > 0\} \cup \{0\}$



is connected [can't be divided into two pieces by disjoint open sets] but not path-connected. [there is no continuous path from 0 to any other point in the set]

Recall: A function f is continuous (on its domain) at z_0 if

~~for all z near~~ for all real $\varepsilon > 0$,

there is a $\delta > 0$,

such that (for all $z \in$ domain of f), ~~if~~ $|z - z_0| < \delta$,

then $|f(z) - f(z_0)| < \varepsilon$.

eg $f(z) = \frac{1}{z}$ is continuous on its domain,

namely $\{ z \in \mathbb{C} \mid z \neq 0 \}$

Fact: $f(z)$ is continuous on its domain [if it's nice, eg open]

if for every open set $D \subseteq \mathbb{C}$,

$f^{-1}(D) = \{ z \in \mathbb{C} \mid \cancel{z \in D}\ z = f(w) \text{ for some } w \in \mathbb{C} \}$

is open (in the domain of f).

The "induced topology" on a set $G \subseteq \mathbb{C}$ has as its open sets the intersections of G with open sets of $\mathbb{C}$.

Next time: derivatives & their properties.