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Absolute values and limits in the complex numbers

Warm-up: $|z| = \sqrt{x^2+y^2}$

$z = x+iy$

$$-\sqrt{x^2+y^2} \le -\sqrt{x^2} \le \underset{\substack{\shortparallel \\ Re(z)}}{x} \le +\sqrt{x^2} \le \sqrt{x^2+y^2}$$

so $\quad -|z| \le Re(z) \le |z|$.

Similarly, looking at $y = Im(z)$, we get $-|z| \le Im(z) \le |z|$.

Cauchy-Schwartz Inequality for real numbers

If $x, y, u, v \in R$, then $(xu+yv)^2 \le (x^2+y^2)(u^2+v^2)$.

proof: $(xu+yv)^2 = (xu)^2 + 2(xu)(yv) + (yv)^2$

$$= x^2u^2 + 2xuyv + y^2v^2$$

Need: $\le$

$$(x^2+y^2)(u^2+v^2) = x^2u^2 + x^2v^2 + y^2u^2 + y^2v^2$$

But $x^2v^2 + y^2u^2 - 2xuyv = (xv)^2 + (yu)^2 - 2(xv)(yu)$

$$= (xv - yu)^2 \ge 0,$$

so $x^2v^2 + y^2u^2 \ge 2xuyv$, as required. //

Corollary: If $z = x+iy$ & $w = u+iv$, then

$$\left(Re(z)Re(w) + Im(z)Im(w)\right)^2 \leq |z|^2|w|^2$$

$$= \left(\frac{z+\bar{z}}{2}\cdot\frac{w+\bar{w}}{2} + \frac{z-\bar{z}}{2i}\cdot\frac{w-\bar{w}}{2i}\right)^2,$$

Triangle Inequality for $\mathbb{C}$ [Prop. 1.6 in the text]

If $z, w \in \mathbb{C}$, then $|z+w| \leq |z|+|w|$.

$z = x+iy$
$w = u+iv$

proof.

$$\begin{aligned}
|z+w|^2 &= |(x+iy)+(u+iv)|^2 \\
&= |(x+u)+i(y+v)|^2 \\
&= (x+u)^2+(y+v)^2 \\
&= x^2+2xu+u^2+y^2+2yv+v^2 \\
&= (x^2+y^2) + 2(xu+yv) + (u^2+v^2) \\
&\leq (x^2+y^2) + 2\sqrt{(xu+yv)^2} + (u^2+v^2) \\
&\leq (x^2+y^2) + 2\sqrt{(x^2+y^2)(u^2+v^2)} + (u^2+v^2) \\
&= |z|^2 + 2\cdot|z||w| + |w|^2 \\
&= (|z|+|w|)^2
\end{aligned}$$

by Cauchy-Schwartz

$\therefore\ |z+w| \leq |z|+|w|$ //

Note that the distance from $z \in \mathbb{C}$ to $w \in \mathbb{C}$
(where we think of $x+iy$ as being the point (x,y))
is given by $|z-w| = \sqrt{(x-u)^2+(y-v)^2}$ where $x+iy = z$, $u+iv = w$.

A "complex function" or "function of a complex variable"
is a function $f: G \to \mathbb{C}$, where $G \subseteq \mathbb{C}$.
G "$\mathrm{dom}(f)$" = "domain of f"

<u>Def'n:</u> $\lim_{z \to z_0} f(z) = w_0$ where f is a complex function,
and $z_0, w_0 \in \mathbb{C}$,
("the limit of $f(z)$ as z approaches z_0 is w_0")
means that for every real numbers $\varepsilon > 0$,
there is a real number $\delta > 0$,
such that if $0 < |z - z_0| < \delta$
then $|f(z) - f(z_0)| < \varepsilon$.

Essentially the same as limits for real numbers,
[most of the same rules work]

... with the caveat that this just like taking limits in $\mathbb{R}^2$, where limits could fail to exist because f different thing depending on how you approach the point where the limit is being taken.

<u>eg</u> $(z = x+iy)$

$$\lim_{z\to 0} \frac{|z|^2}{Re(z)\,Im(z)} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} \frac{x^2+y^2}{xy} = \lim_{x\to 0} \frac{x^2+x^2}{x^2} = 2$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=-x}} \frac{x^2+y^2}{xy} = \lim_{x\to 0} \frac{x^2+(-x)^2}{x(-x)} = \lim_{x\to 0} \frac{2x^2}{-x^2} = -2$$

$\therefore \displaystyle\lim_{z\to 0} \frac{|z|^2}{Re(z)\,Im(z)}$ does not exist.

We can now define continuity:

$f(z)$ is <u>continuous at z_0</u> if $\lim_{z \to z_0} f(z) = f(z_0)$.

$f(z)$ is <u>continuous</u> if it is continuous ~~for~~ at all $z_0 \in \mathrm{dom}(f)$.

... and derivatives:

We'll be assuming (usually) that $f(z)$ is defined for all z near z_0.

The <u>derivative of $f(z)$ at z_0</u> is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$(h \in \mathbb{C}) \qquad = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

provided that this limit exists.

f is said to be be <u>differentiable at z_0</u> if it has a derivative at z_0.

The basic properties (eg linearity) of derivatives are the same as for reals.

If $f(z)$ is differentiable at all z near z_0 (i.e. for all z so that for some real $\delta > 0$, $|z - z_0| < \delta$),

then $f(z)$ is said to be <u>holomorphic</u> (or <u>analytic</u>), at z_0.

[If $f(z)$ is defined and holomorphic at all complex numbers, then $f(z)$ is said to be <u>entire</u>.

<u>Q.</u>: If $f(z)$ is defined & diff'ble at all $z \in \mathbb{C}$, then $f(z)$ is entire. Why?]

Please see Examples 2.8 & 2.9 in the textbook.

Ex. 2.9: $f(z) = \bar{z}$ is <u>not</u> differentiable anywhere

Ex. 2.8: $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = (\bar{z})^2 = \bar{z} \cdot \bar{z}$ is differentiable at 0, but <u>only</u> at 0.