2022-01-11

The Basics of the Complex Numbers

The complex numbers, $\mathbb{C}$, are what you get when you add ~~a~~ a <u>square</u> root of -1, $i=\sqrt{-1}$, to the real numbers, and then do arithmetic as usual...

$$\mathbb{C} = \{ x+iy \mid x,y \in \mathbb{R} \} \qquad (\text{and } i^2=-1).$$

Why do we use them? We <u>need</u> them...

<u>c</u>. 1545 Girolamo Cardano published a complete solution to the cubic & quartic <u>equations</u> in his <u>Ars Magna</u> and had to consider the <u>square</u> roots of negative numbers to get complete solutions.

<u>c</u>. 1572 Rafael Bombelli in his <u>L'Algebra</u> was the first to treat the complex numbers as a number <u>system</u> in its own right.

In 1637, René Descartes introduced the terminology that numbers of the form iy are "imaginary". (Basically, he viewed them as unreal, but useful.)

Note that $\sqrt{-1}\cdot\sqrt{-1} = -1$ seems to violate the familiar rule that $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$ [This only works for $a, b \geq 0$.] because then we'd have $\sqrt{-1}\cdot\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$.

The idea that complex numbers can be identified with points in the Cartesian plane, $(x,y) \Leftrightarrow x+iy$, seems to have occurred to several people independently c. 1800. One of them was Gauss, who established the modern notation for complex numbers and proved the Fundamental Theorem of Algebra. [If $p(z)$ is a polynomial with complex coefficients, then $p(z)$ can be factored into linear factors (with complex coefficients).]

Basics: 1° The conjugate of the complex number

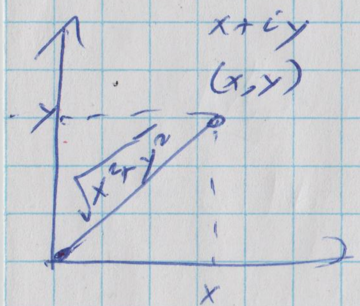
$z = x + iy$ is $\bar{z} = \overline{x+iy} = x - iy$.

2° The real and imaginary parts of $z = x + iy$

are $Re(z) = x$ and $Im(z) = y$.

Note that: $Re(z) = \dfrac{z+\bar{z}}{2}$

and $Im(z) = \dfrac{z-\bar{z}}{2i}$.



3° The modulus or absolute value of $z = x + iy$

is $|z| = \sqrt{x^2+y^2}$.

Note that: $z \cdot \bar{z} = (x+iy)(x-iy)$

$= x^2 - ixy + ixy - i^2y^2$

$= x^2 - (-1)y^2 = x^2+y^2, \geq 0$

so $|z|^2 = z \cdot \bar{z}$ and $|z| = \sqrt{z \cdot \bar{z}}$.

4° The reciprocal ~~inverse~~ of $z = x+iy \neq 0$ is

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\cdot\bar{z}}$$

Note that the conjugate ~~of a ...~~ plays well with arithmetic in the complex numbers,

es $\overline{w+z} = \bar{w} + \bar{z}$, $\overline{wz} = \bar{w}\bar{z}$, $\overline{w/z} = \bar{w}/\bar{z}$.

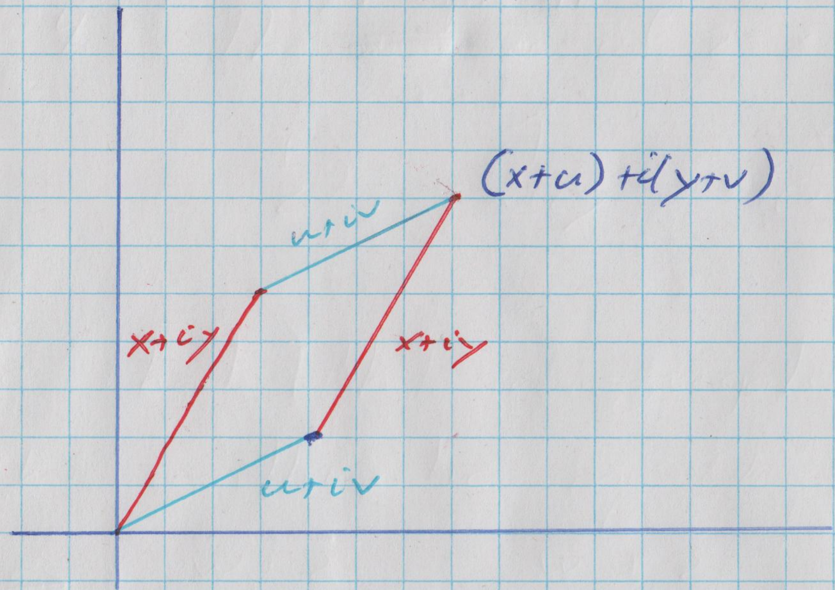
We can divide complex numbers (other than 0) via $w/z = w\cdot\frac{1}{z} = \frac{w\cdot\bar{z}}{z\cdot\bar{z}}$.

The <u>Complex Plane</u> is the identification of the complex number system with the Cartesian plane via

$$(x,y) \Longleftrightarrow x+iy.$$

This lets us visualize various operations and, in particular, relate them to the linear algebra of $\mathbb{R}^2$.

eg



We want to add $z = x + iy$ to $w = u + iv$

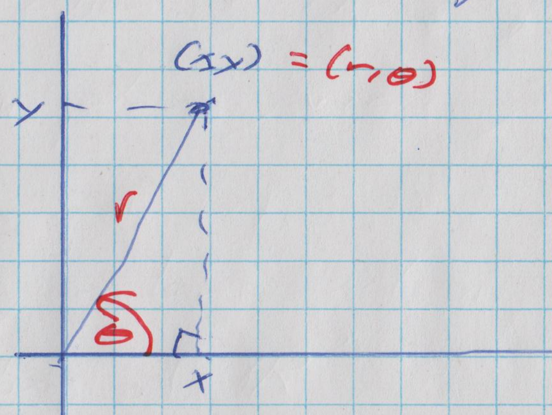
ie addition of complex numbers can be visualized as vector addition

What about multiplication of complex numbers?

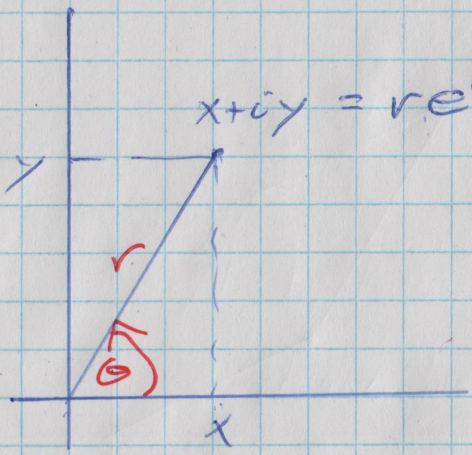
This is better handled by giving the complex plane polar coordinates:

(r, θ) in polar coordinates corresponds to (x, y) in Cartesian coordinates via:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$r = \sqrt{x^2 + y^2}$$



Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$



$$x+iy = re^{i\theta} = r\cos(\theta) + ir\sin(\theta) \qquad (r \geq 0)$$

$$\left(= r\cos(\theta + k2\pi) + ir\sin(\theta + k\cdot 2\pi) \right.$$
$$\left. \text{for all integers } k \right)$$

$$\tan(\theta) = \frac{y}{x}$$

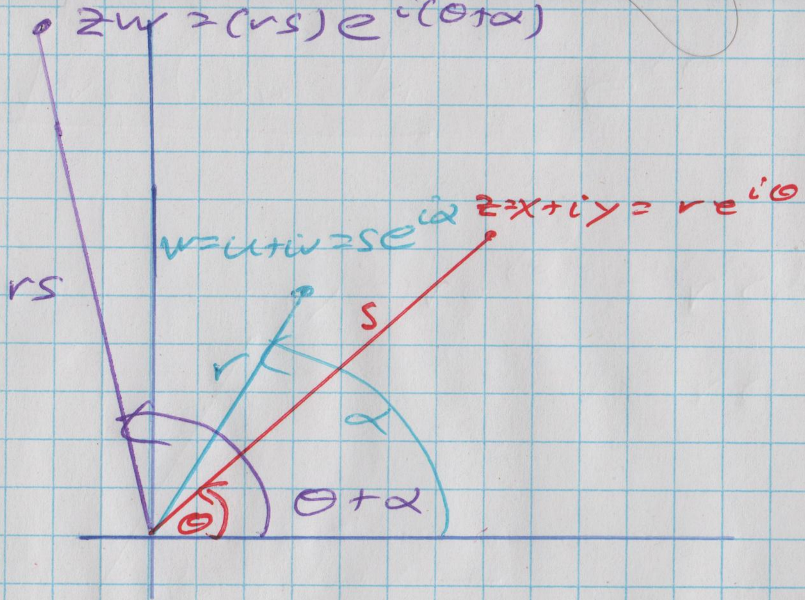
$$r = \sqrt{x^2+y^2} = |z|$$

$$\theta = \arctan\left(\frac{y}{x}\right) \qquad (\text{as long as } x \,\&\, y > 0)$$

[We'll come back to the ambiguity & limitations of doing things this way as we encounter problems.]

This does make it a little more natural to handle complex multiplication: $z = re^{i\theta}$ & $w = se^{i\alpha}$

$$\Rightarrow zw = (re^{i\theta})(se^{i\alpha})$$
$$= rse^{i(\theta+\alpha)}$$



$$zw = (re^{i\theta})(se^{i\alpha})$$
$$= (rs)\, e^{i(\theta+\alpha)}$$

$$zw = (x+iy)(u+iv)$$
$$= (xu - yv)$$
$$+ i(xv + yu)$$

<u>De Moivre's Thm:</u> $\left(\cos(\theta) + i\sin(\theta)\right)^n$

$$= \cos(n\theta) + i\sin(n\theta)$$