Mathematics 3260H – Geometry II: Projective and non-Euclidean geometry
TRENT UNIVERSITY, Winter 2015

Take-home Final Exam
Due on Wednesday, 22 April, 2014.

Instructions: Do all three of parts $\bigcirc$, $\between$, and $\triangle$, and, if you wish, part $\square$ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, you may not consult any other sources, nor consult or work with any other person on this exam.

Part $\bigcirc$. Do *all* four (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Suppose $\triangle DEF$ in the elliptic plane has $\angle DEF = \dfrac{\pi}{2}$. Show that $\cos(|DF|) = \cos(|DE|)\cos(|EF|)$.

2. Suppose γ is a collineation of a projective plane which is its own inverse, *i.e.* $\gamma^2 = i$ is the identity map. Show that γ must fix more than one point and more than one line.

3. $\triangle ABC$ in the hyperbolic plane has $|BC| = 1$ and $\angle ABC = \angle ACB = \dfrac{\pi}{3}$. Compute $|AB|$, $|AC|$, $\angle BAC$, and the area of the triangle.

4. Find as many essentially different examples of incidence structures that satisfy axioms **I** and **II** for a projective plane, but not axiom **III**.

Part $\between$. Do any *two* (2) of problems **5** – **8**. *[20 = 2 × 10 each]*

5. Using Euclid's Postulates I-IV, show that the sum of the interior angles of any triangle is π if and only if the sum of the interior angles of any quadrilateral is is 2π.

6. Determine whether the Side-Angle-Angle congruence criterion for triangles is true in each of the elliptic, Euclidean, and hyperbolic planes.

7. Is it possible to tile the elliptic plane with a triangle? (That is, cover all of the elliptic plane with congruent copies of the same triangle, with no overlap between the copies except for the sides and vertices.) Give an example of such a tiling or explain why there can't be one.

8. Explain why Desargues' Theorem is true in the elliptic plane. Is it also true in the hyperbolic plane?

Part $\triangle$. Do any *two* (2) of problems **9** – **12**. *[20 = 2 × 10 each]*

9. Show that Desargues' Theorem is sometimes, but not always, true in the Moulton plane.

10. $\mathbb{Z}_3$ is the field with three elements, *i.e.* $\mathbb{Z}_3 = \{0, 1, 2\}$, with $+$ and $\cdot$ done modulo 3. Determine whether the projective plane with $\mathbb{Z}_3$ as its underlying algebraic structure has a (P, ℓ)-collineation for some P not incident with ℓ or not.

11. Suppose δ is a collineation of a projective plane $\Pi = (\mathcal{P}, \mathcal{L}, I)$ that fixes all four vertices of quadrangle $ABCD$, but is not the identity. Show that the collection of all points and lines of Π that are fixed by δ, with incidence between them as in Π, is itself a projective plane.

12. Suppose one replaces axiom **AII** of an affine plane (*i.e.* Playfair's Axiom) with its hyperbolic counterpart:

AII*. Given any line ℓ and any point P not on ℓ, there is more than one line through P which is parallel to ℓ.

Give a finite example of an incidence structure satisfying **AI**, **AII***, and **AIII**, and with every line having the same number of points on it, or show that there can be no such structure.

Part □.

■. Write a poem about geometry or mathematics in general. *[2]*

[Total = 80]

I HOPE THAT YOU ENJOYED THIS COURSE.
MAY THE SUMMER BE EVEN BETTER!