Mathematics 326H – Geometry II: Projective and non-Euclidean geometries
TRENT UNIVERSITY, Winter 2007

Problem Set #6
Due on Thursday, 5 April, 2007.

Free completion

In this problem set we consider a standard method for constructing projective planes:

Suppose $\mathcal{C}_0 = (\mathcal{P}_0, \mathcal{L}_0, I_0)$ is a configuration consisting of a set of points, $\mathcal{P}_0$, a set of lines, $\mathcal{L}_0$, and a relation of incidence, I_0, specifying which points are incident with which lines. For each $n \geq 0$, given the configuration $\mathcal{C}_n = (\mathcal{P}_n, \mathcal{L}_n, I_n)$, define the configuration $\mathcal{C}_{n+1} = (\mathcal{P}_{n+1}, \mathcal{L}_{n+1}, I_{n+1})$ as follows:

- $\mathcal{P}_{n+1}$ includes all the points of $\mathcal{P}_n$, together with a new and distinct point of intersection for every pair of lines in $\mathcal{L}_n$ which do not already have a common point of intersection in $\mathcal{C}_n$.
- $\mathcal{L}_{n+1}$ includes all the lines of $\mathcal{L}_n$, together with a new and distinct line joining each pair of points in $\mathcal{P}_n$ which do not already have a line joining them in $\mathcal{C}_n$.
- I_{n+1} is I_n, plus the incidences involving the added points and lines described above.

Finally, let $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$, $\mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n$, and $I = \bigcup_{n=0}^{\infty} I_n$. The incidence structure $\mathcal{C} = (\mathcal{P}, \mathcal{L}, I)$ is the *free completion* of the configuration $\mathcal{C}_0 = (\mathcal{P}_0, \mathcal{L}_0, I_0)$.

1. Show that the free completion of an arbitrary (non-empty) configuration must satisfy axioms **I** and **II** for a projective plane. *[5]*

2. Give an example of a (non-empty) configuration whose free completion is *not* a projective plane. *[1]*

3. Assume $\mathcal{C}_0$ consists of four points and no lines. Show that the free completion $\mathcal{C}$ of $\mathcal{C}_0$ is a projective plane. *[4]*