Trent University

Mathematics 326H – Geometry II: Projective and non-Euclidean geometries

Take-home Final Examination

Due: 25 April, 2007

Instructions: Give complete answers to receive full credit, including references to any and all sources you used. You may use your texts from this and any other courses, as well as any handouts, class notes, and the like; you may also ask the instructor to clarify the instructions or any of the questions; and you may use a calculator or computer to perform any necessary calculations. *You may not consult any other sources, nor give or receive any other aid on this exam, except with the intructor's explicit permission.*

Part I – Axiomatics. Do any *two* of **1** – **3**. *[20 = 2×10 each]*

1. Find as many essentially different examples of incidence structures that satisfy axioms **I** and **II** for a projective plane, but not axiom **III**.

2. Show that Euclid's Fifth Postulate is equivalent to the Triangle Axiom.

3. It was noted in class that the spherical model of the elliptic plane is (an instance of) the real projective plane. It was also noted that if one removes a line from the real projective plane, one is left with the real affine plane, *i.e.* the Euclidean plane. Thus the Euclidean plane is embedded in a model of the elliptic plane, even though these satisfy very different axioms. Explain why this isn't a problem.

Part II – Non-Euclidean geometries. Do any *two* of **4** – **6**. *[20 = 2×10 each]*

4. Suppose k, ℓ, and m are three distinct lines in the Poincaré model such that k is perpendicular to both ℓ and m. Are ℓ and m necessarily parallel or not? Either way, prove it.

5. Let ℓ be a line of the hyperbolic plane, A a point on ℓ, and AB a line segment perpendicular to ℓ. As we slide A along ℓ, keeping $|AB|$ constant, B sweeps out a curve in the hyperbolic plane. Prove that this curve is not a line of the hyperbolic plane.

6. Show that every line of the elliptic plane (not just those in the spherical model) must have finite length.

Part III – Projective geometry. Do any *three* of **7** – **10**. *[30 = 3×10 each]*

7. Suppose γ is a collineation of a projective plane which is its own inverse, *i.e.* $\gamma^2 = \gamma \circ \gamma$ is the identity map. Show that γ must fix more than one point and more than one line.

8. Draw (all of) a projective plane which has four points on each line.

9. Does the collineation of the real projective plane (using projective coordinates) induced by a 3×3 matrix $\mathbf{A}$ have to fix some point or not? Either way, prove it.

10. Suppose $\Pi = (\mathcal{P}, \mathcal{L}, I)$ is a projective plane and $\ell \in \mathcal{L}$ is one of its lines. Define the incidence structure $\mathcal{A} = (\mathcal{P}^*, \mathcal{L}^*, I^*)$ as follows:

i. $\mathcal{P}^*$ includes every point in $\mathcal{P}$ except those which are on the line ℓ.
ii. $\mathcal{L}^*$ includes every line in $\mathcal{L}$ except ℓ.
iii. I^* is the relation I restricted to the points in $\mathcal{P}^*$ and the lines in $\mathcal{L}^*$.

Show that $\mathcal{A} = (\mathcal{P}^*, \mathcal{L}^*, I^*)$ is an affine plane.

[Total = 70]

Part IV - Fun. Bonus!

$\circ^{\circ^{\circ}}$**.** Write a limerick touching on geometry or mathematics in general. *[2]*

Sylvester's Theorem
(The Rank-Nullity Law)

A mathematician, Sylvester,
Had a wife he would often pester,
"As I raised the rank
All my null spaces shrank."
"Add them," she said, so he kissed her.

I HOPE YOU ENJOYED THE COURSE. ENJOY THE SUMMER!