Axioms for Affine and Projective Planes

The geometrical notion that we will focus on, while discarding notions like distance and angle, is that of *incidence*, *i.e.* the relation of points being on lines or lines passing through points.

DEFINITION. An *incidence structure* is a triple $(\mathcal{P}, \mathcal{L}, \mathbf{I})$, consisting of a set $\mathcal{P}$ of *points*, a set $\mathcal{L}$ of *lines*, and a relation $\mathbf{I}$ of *incidence* between elements of $\mathcal{P}$ and elements of $\mathcal{L}$.

If a point $P \in \mathcal{P}$ is *incident* with a line $\ell \in \mathcal{L}$, *i.e.* $P\mathbf{I}\ell$, P is usually said to be *on* ℓ, and ℓ is usually said to *pass through* P. Two lines which are both incident with a particular point are usually said to *intersect*, or to be *coincident*, at that point. Two points which are both incident with the same line are sometimes said to be *joined* or *connected* by that line.

Note that the relation of incidence in a general incidence structure may be completely arbitrary. Points may or may not be connected by lines, lines may or may not intersect, there may be multiple points of intersection for a given pair of lines, and there may be multiple lines joining a given pair points. We will almost always stick to incidence structures which avoid some of these pathologies:

DEFINITION. A *configuration* is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathbf{I})$ satisfying the following conditions:

i. Any two distinct points are incident with at most one line.
ii. Any two distinct lines are incident with at most one point.

Note that some geometries, such as the geometry of points and great circles on the surface of a sphere, are not configurations. Two different great circles of a sphere will always intersect in two points, while two points on the sphere that are directly opposite each other through the centre of the sphere will be connected by infinitely many great circles.

The incidence structure most likely to turn up in basic geometry is the Euclidean plane, which is an example of an affine plane. Affine planes are of interest to us because of their close connections to projective planes. A preview: if you take a away one line and all the points on it from a projective plane, you get an affine plane. Going the other way, if you add a single line "at infinity" to a an affine plane, with the points on it being where each class of parallel lines meet, you get a projective plane.

DEFINITION. An *affine plane* is a configuration $(\mathcal{P}, \mathcal{L}, \mathbf{I})$ satisfying the following axioms:

AI. Any two distinct points are incident with an unique line.
AII. Given a point P and a line ℓ not incident with P, there is an unique line m incident with P which has no point in common with ℓ.
AIII. There exist three points which are not incident with the same line.

Finally, here are the axioms for projective planes:

DEFINITION. A *projective plane* is a configuration $(\mathcal{P}, \mathcal{L}, \mathbf{I})$ satisfying the following axioms:

I. Any two distinct points are incident with an unique line.
II. Any two distinct lines are incident with an unique point.
III. There exist four points such that no three of them are incident with the same line.