Trent University, Fall 2021

**Mathematics 3260H – Geometry II: Projective and Non-Euclidean Geometry**

Take-home Final Examination

*Due on Friday, 17 December, on paper or via Blackboard.*

**Instructions:** Do all three of Parts **A**, **B**, and **C**, and, if you wish Part **D**. Give complete answers to receive full credit, including references to any and all sources you used. You may use your texts from this and any other courses, as well as any handouts, class notes, and the like; you may also ask the instructor to clarify the instructions or any of the questions; and you may use a calculator or computer to perform any necessary calculations. *You may not consult any other sources, nor give or receive any other aid on this exam, except with the intructor's explicit permission.*

**Part A**argh. Do any *three* (3) of **1** – **4**. *[30 = 3×10 each]*

**1.** Consider the following geometry $\mathcal{G}$:

*i.* The points of $\mathcal{G}$ are the points on the cone $z = \sqrt{x^2 + y^2}$ in $\mathbb{R}^3$.

*ii.* The lines of $\mathcal{G}$ are the intersections of the cone with planes in $\mathbb{R}^3$ that contain the $z$-axis.

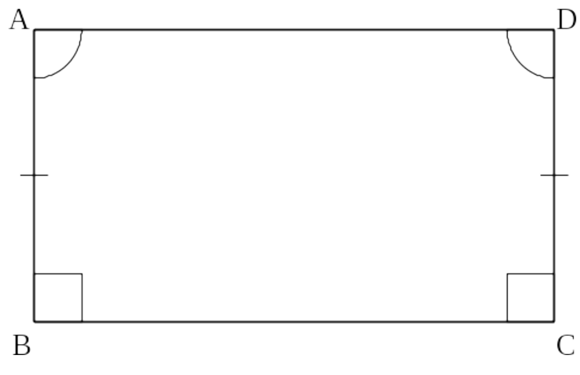
*iii.* Distances between points on the cone are measured along the shortest line of $\mathcal{G}$ that connects the two points.

*iv.* Angles between intersecting lines of $\mathcal{G}$ are the angles between the corresponding planes.

Which of Euclid's Postulates I-IV, along with Playfair's Axiom, are true in $\mathcal{G}$?

**2.** Suppose $\Pi = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a projective plane. Define the incidence structure $\Pi' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ as follows: *(i)* $\mathcal{P}' = \mathcal{L}$, *(ii)* $\mathcal{L}' = \mathcal{P}$, and *(iii)* for all $p \in \mathcal{P}'$ and $L \in \mathcal{L}'$, $p\,\mathbf{I}'L \iff L\,\mathbf{I}\,p$. Show that $\Pi' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ is also a projective plane.

**3.** Recall from Assignment #11 that a *Saccheri quadrilateral* is a quadrilateral $ABCD$ in which sides $AB$ and $CD$ are perpendicular to the base $BC$, with $A$ and $D$ on the same side of $BC$, and with $AB = CD$ (*i.e.* $AB$ and $CD$ have the same length).




Assuming Postulates I-IV (and A and S, if needed), show that if every Saccheri quadrilateral is a rectangle, then Postulate V holds. (You may use Playfair's Postulate instead of Euclid's Postulate V.)

*Sadly, there are more problems on page 2.*

**4.** Explain why one can construct, in the elliptic plane, a model of the hyperbolic plane.

**Part Boo.** Do any *two* (2) of **5** – **8**. *[20 = 2×10 each]*

**5.** Suppose $\gamma$ is a collineation of a projective plane which is its own inverse, *i.e.* $\gamma^2 = \gamma \circ \gamma$ is the identity map. Show that $\gamma$ must fix more than one point and more than one line.

**6.** Determine whether the projective plane coordinatized by $\mathbb{Z}_5$ has any $(P, \ell)$-central collineation, other than the identity collineation, for some point $P$ and line $\ell$ such that $P$ is not incident with $\ell$.

NOTE: $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, with addition and multiplication done modulo 5.

**7.** Show that any collineation of the real projective plane (using projective coordinates) induced by an invertible $3 \times 3$ matrix $\mathbf{P}$, as in Assignment #6, has to fix at least one point.

**8.** Suppose that a projective plane is coordinatized by the ternary ring $(R, T)$ relative to the quadrangle $OIUV$, so $O = (0, 0)$, $I = (1, 1)$, $U = (0)$, and $V = (\infty)$. Show that if the plane is $(U, OV)$-transitive, then $(R, T)$ is linear.

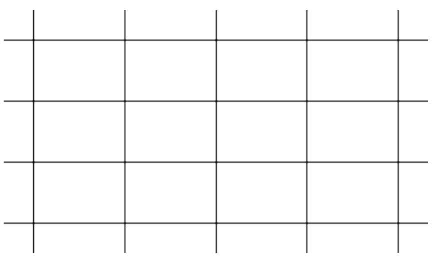
**Part Cough.** Do any *two* (2) of **9** – **12**. *[20 = 2×10 each]*

**9.** Pick one of the elliptic or hyperbolic planes to work in. Let $\ell$ be a line of the plane, $A$ a point on $\ell$, and $AB$ a line segment perpendicular to $\ell$. As we slide $A$ along $\ell$, keeping the length of $AB$ constant, $B$ sweeps out a curve in the plane. Show that this curve is not a line of the plane.

**10.** Suppose $\triangle ABC$ has $\angle ABC = \angle ACB = \dfrac{\pi}{3}$ and its side $BC$ has length 1. Compute the area of $\triangle ABC$ in the ...

**a.** Euclidean plane. *[2]* **b.** elliptic plane. *[4]* **c.** hyperbolic plane. *[4]*

**11.** A shape is said to *tile* a plane if one can completely cover the plane with congruent copies of the shape, with no overlap between the copies except at their borders. For example, the following is (part of) a tiling of the Euclidean plane by a rectangle:



**a.** Show that one can tile the Euclidean plane with an equilateral triangle. *[1]*
**b.** Show that one can tile the elliptic plane with an equilateral triangle. *[4]*
**c.** Determine whether it is possible to tile the hyperbolic plane with an equilateral triangle. Give an example of such a tiling or explain why there can't be one. *[5]*

*Even more sadly, there are still more problems on page 3.*

**12.** Suppose $k$, $\ell$, and $m$ are three distinct lines in the hyperbolic plane such that $k$ meets both $\ell$ and $m$ – at different points! – at right angles. Determine whether $\ell$ and $m$ must be parallel to each other.

*[Total = 70]*

**Part Duh.** Bonus!

**13.** Write a poem touching on projective or non-Euclidean geometry. *[1]*

I HOPE THAT YOU ENJOYED THE COURSE. HAVE A GOOD BREAK!