Mathematics 3260H – Geometry II: Projective and Non-Euclidean Geometry

TRENT UNIVERSITY, Fall 2021

Assignment #6

Collineations Meet Linear Algebra

Due on Friday, 22 October.

*May be submitted on paper or via Blackboard.**

Recall from class that we could give the real projective plane a coordinate system derived from its presentation as having the points of the plane be the lines through the origin in $\mathbb{R}^3$ and the lines of the plane be the planes through the origin, with incidence being given by inclusion. In this scheme, the coordinates of the points are direction vectors of the corresponding lines through the origin in $\mathbb{R}^3$ and the coordinates of the lines are normal vectors of the corresponding planes through the origin in $\mathbb{R}^3$. A point and line are then incident exactly when the dot product of their coordinates is 0.

To be explicit about how these *projective* (also called *homogeneous*) coordinates work:

- Points of the real projective plane are given by vectors $(u, v, w) \in \mathbb{R}^3$, where u, v, and w are not all 0, and if $(u, v, w) = \lambda(x, y, z)$ for some $\lambda \neq 0$, then (u, v, w) and (x, y, z) denote the same point of the plane.
- Lines of the real projective plane are given by vectors $[a, b, c] \in \mathbb{R}^3$, where a, b, and c are not all 0, and if $[a, b, c] = \lambda[d, e, f]$ for some $\lambda \neq 0$, then $[a, b, c]$ and $[d, e, f]$ denote the same line of the plane.
- The point with coordinates (u, v, w) is incident with the line with coordinates $[a, b, c]$ if and only $(u, v, w) \cdot [a, b, c] = ua + bv + cw = 0$.

Note that since we are thinking of them as coordinates, we usually write the vectors in question as row vectors instead of column vectors. In addition, we distinguish points from lines by using parentheses for the coordinates of points and brackets for the coordinates of lines.

The questions below look at the connection between linear transformations of $\mathbb{R}^3$ and the collineations of the real projective plane.

1. Suppose $\mathbf{P}$ is an invertible 3×3 matrix over the real numbers. Define a function φ from points to points of the real projective plane by $\varphi(\mathbf{p}) = \mathbf{pP}$, where $\mathbf{p}$ denotes a coordinate vector of the point P of the plane. Show that there is an invertible 3×3 matrix $\mathbf{L}$ such that if we extend φ to a function from lines to lines of the real projective plane by $\varphi(\boldsymbol{\ell}) = \boldsymbol{\ell}\mathbf{L}$, where $\boldsymbol{\ell}$ denotes a coordinate vector of the line ℓ of the plane, then φ is a collineation of the real projective plane. *[10]*

Note that since we are using row vectors, we have to put the matrix on the right instead of the left for the matrix multiplication to make sense.

2. *Bonus!* Is the converse of the result in question **1** true? That is, is every collineation of the real projective plane given by some linear transformations *i.e.* matrices) of $\mathbb{R}^3$ as in question **1**? *[5]*

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* All else failing, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`