Another way to build a projective plane

So far we have two methods:
1) Start with an affine plane and add a "line at infinity".
2) Start with an algebraic structure (like a field) and do the linear algebra trick.

3) Free completion

· Start with four points, with no three on a line, or four lines, with no three meeting at a point, and then add what you need inductively.

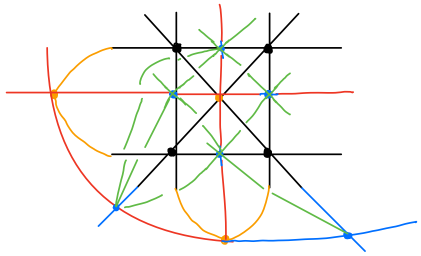
0. We'll start with four points and no lines
$\mathcal{P} = \{P_1, P_2, P_3, P_4\}$, $\mathcal{L} = \emptyset$, and no lines means incidence doesn't happen.

1. We add six lines, one to join each pair of points.
$\mathcal{P} = \{P_1, P_2, P_3, P_4\}$, $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_5, \mathcal{L}_6\}$
I: $P_1 I \mathcal{L}_1$, $P_2 I \mathcal{L}_1$, $P_1 I \mathcal{L}_2$, $P_3 I \mathcal{L}_2$, $P_1 I \mathcal{L}_3$, ...

2. We add points, one to join each pair of lines that do not already intersect.

⋮

n. We add new lines for each pair of points not already connected, and add new points for each of lines not already intersected.



Steps 0, 1, 2, 3, 4, 6

Claim: This process, starting with four points and no lines, produces a projective plane, after infinitely many steps.

Proof: We need to check that the resulting structure satisfies Axioms I – III.

I. Suppose we have points P & Q in the final structure. At the earliest stage n such that both points were present, then either one was added to a line the other was aready on, so they are connected, or, if this was not the case, they connected at stage n+1.

II. Suppose $\mathcal{L}$ and m are both first present at stage n. If they are not already intersecting, then they will get a common point at stage n+1.

III. The four points we started with had no lines between them (Step 0). At Step 1 we added lines to join these points two at a time. These lines remain distinct through the rest of the process, which ensures that no lines intersect more than once. No line can be added or modified to include three of the beginning points.

Next: Reverse-engineering affine-style coordinates from the geometry of a projective plane.