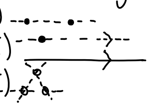
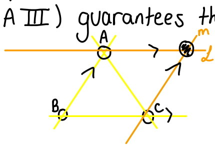
Some easy proofs of Simple facts about projective (and affine) planes.

Proposition: Every affine plane has at least four points

(AI)
(AII)
(AIII)



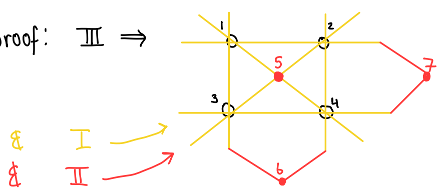
proof: (AIII) guarantees there are 3 points (not all on the same line)



- (AII) guarantees that there is a line through one of the three points parallel to the one connecting the other two.
- Use (AII) again to find a line through one of the othe two points parallel to the line formed by the remaining two points.
- $\mathcal{l}$ is the only line through A that is parallel to BC.
- Conversely, BC is the only line through C that is parallel to $\mathcal{l}$
- m is the only line through C parallel to AB
- Conversely, AB is the only line through A parallel to m.
- If m and $\mathcal{l}$ did not intersect they would be parallel, violating uniqueness for existing parallels.
- Thus, $\mathcal{l}$ and m must intersect at some point. It can't be B without violating the uniqueness in (AI).
- Thus, since they also can't intersect in A or C, they must intersect in a fourth point.

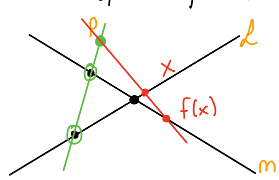
Proposition: A projective plane must have at least seven points.

proof: III ⇒



Proposition: Every line of a projective plane has just as many points as every other line of that plane.

Proof:



(II)

- The two lines intersect at one point.
- We need to make a 1-1 correspondence between the remaining points on the lines
- Pick a point on each of the two lines (other than the intersection) & connect them with a line.
- If every line has at least 3 points, then there is a third one on this new line.
- We pair the points on $\mathcal{l}$ with the points on m by associating a point X on $\mathcal{l}$ with $f(x) = m \cap PX$.
- The uniqueness in (I & II) guarantees that if $x \neq Y$ for x & Y on $\mathcal{l}$, then $f(x) \neq f(Y)$, So f is 1-1. You car run this in reverse, so f is also onto.
- Thus $\mathcal{l}$ and m have the same number of points.