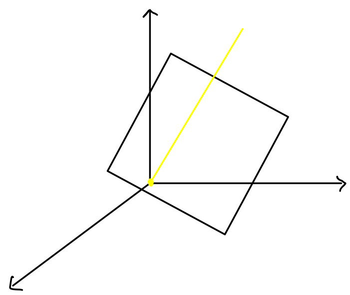
Examples of Projective Planes

1° one way to get what amounts to the real projective plane.
The "points" are the lines through the origin in $\mathbb{R}^3$.
The "lines" are the planes through the origin in $\mathbb{R}^3$.
Incidence is inclusion:
→ if the line ("point") is in the plane ("line") then they are incident.



We can turn this into linear algebra:
Represent the "points" by the direction vectors of the lines $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Represent the "lines" by the normal vectors of the planes $\begin{bmatrix} d \\ e \\ f \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The plane ("line") represented by $\begin{bmatrix} d \\ e \\ f \end{bmatrix}$ is incident with the line ("point") represented

$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ if $\begin{bmatrix} d \\ e \\ f \end{bmatrix} \underset{\text{dot product}}{\bullet} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

We claim this gives a projective plane. We need to show that it satisfies the axioms.

Axiom I: Any two points are connected by an unique line.
Given two different points, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$. We need a normal vector for the plane, ie a vector which has a dot product of $\emptyset$ with both of these.
[ie perpendicular to both].

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \underset{\text{cross product}}{\times} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ works!

Axiom II: Any two different lines intersect at a unique point.
Suppose we have the lines represented by $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$. We need a line,
ie a vector which has a dot product of $\emptyset$ with both of these.
[ie perpendicular to both].

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ works!

Axiom III: There exist four points such that no three are on the same line.
That is, there are 4 non-zero vectors $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ such that there is no vector $\vec{n} \neq \vec{0}$ such that $\vec{n} \cdot \vec{a} = 0 = \vec{n} \cdot \vec{b}$
$= \vec{n} \cdot \vec{c}$
$= \vec{n} \cdot \vec{d}$

Can we find such vectors?
Sure, let $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Note: We can execute this algebraic definition for basically any field, and some things that aren't fields.
$\mathbb{Z}_3$ (integers mod 3)
$\mathbb{Q}$
$\mathbb{C}$
$\mathbb{H}$ - quaternions (not quite a field as $a \cdot b \neq b \cdot a$ some of the time)
$\hookrightarrow = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$
where $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$, & $i^2 = j^2 = k^2 = -1$

2. We'll construct the "cartesian" affine plane for $\mathbb{Z}_3$ (integers mod 3) and then extend it to a projective plane.
$\mathbb{Z}_3 = \{0, 1, 2\}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

We'll start with $\mathbb{Z}_3^2$