Transitivity & Ternary Rings Continued

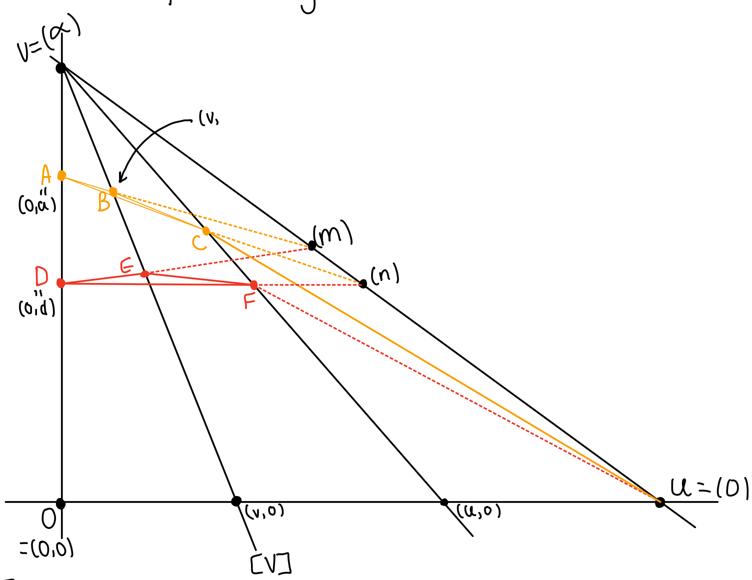
Algebraic Preliminaries

- A loop is a set L with a binary operation '$\circ$' such that
  (1) For all $a,b \in L$ there is a unique $x \in L$ s.t. $a \circ x = b$,
  (2) For all $a,b \in L$ there is a unique $y \in L$ s.t $y \circ a = b$,
  and (3) there is an identity element ie an $e \in L$ s.t for all $a \in L$
  $a \circ e = e \circ a = a$.
- If (R, T) is a ternary ring (like those we get for the coordinates in a proj. plane)
  (T: ternary operation)
  and we define $+$ and $\cdot$ on R by $a+b = T(1,a,b)$ and $a \cdot b = T(a,b,0)$, then $(R,+)$ is a loop with identity 0, and $(R \setminus \{0\}, \cdot)$ is a loop with identity 1.
  proof: Follows from the def'n of what a ternary ring is.

Theorem: Suppose $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ ($\mathcal{I}$: Incidence) is a projective plane, and we set up coordinates using $O = (0,0)$, $I = (1,1)$, $U = (0)$, $V = (\infty)$ and let (R, T) be the corresponding ternary ring.

- Then T is "linear" ie $T(m,x,b) = (m \cdot x) + b$
  $= T(1, T(m,x,0), b)$

  iff whenever we have two triangles ABC and DEF such that
  (i) They are in perspective from $V = (\infty)$
  (ii) A and D are on $OV = [0]$
  (iii) $AB \cap DE$ and $AC \cap DF$ are both on $UV = [\infty]$
  and (iv) BC is incident with $U = (0)$,
  we also have
  (v) EF passes through $U = (0)$ as well.



$\boxed{\Leftarrow}$ Assume we satisfy the triangle conditions (for all such pairs of triangles)

- Suppose $A = (0,a)$ & $D = (0,d)$ while VBE & VCF intersect $OU = [0,0]$ in $(v,0)$ and $(u,0)$ respectively and $AB \cap DE = (m)$ and $AC \cap DF = (n)$
- Then $B = (v, T(m,v,a))$ since $AB = [m,a]$ and B is on $[v]$.
- Similarly, $C = (u, T(m,u,a))$,
  $E = (v, T(m,v,d))$,
  and $F = (u, T(n,u,d))$.

- Since BC passes through $u = (0)$ so $T(m,v,a) = T(n,u,a)$ (n) and this is true for all a.
- By our hypothesis, EF also passes through (0), so $T(m,v,d) = T(n,u,d)$ (for all d)
- Recall: We want to show that $T(m,x,b) = (x \cdot m) + b = T(1, T(x,m,0), b)$
- In particular, since $T(m,x,0) = T(1, T(m,x,0), 0)$ so $T(m,x,b) = T(1, T(m,x,0), b)$

$a = 0 \;, d = b$

$\boxed{\Longrightarrow}$ Run the other direction in reverse, more or less.

Corollary: If a plane is $((\infty), [\infty])$ transitive (equivalently, the $((\infty),[\infty])$-Desargues Thm holds), the ternary ring is "linear".