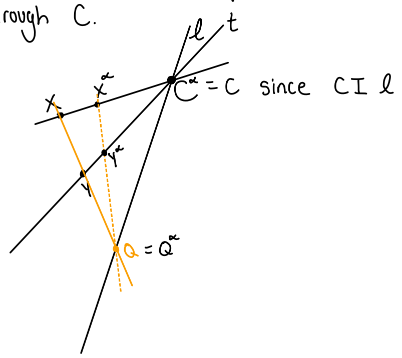
Back to showing that a collineation is central if it is axial (& vice versa)

α is axial means it has an <u>axis</u>, <u>ie</u> a line ℓ s.t. $P^\alpha = P$ for every $P \, I \, \ell$

<u>Case 1</u>: There is a point C not on ℓ with $C^\alpha = C$. Then C is a centre for α (last time)

<u>Case 2</u>: For every point X not on ℓ we have $X^\alpha \neq X$.

We need to find a point $C \, I \, \ell$ s.t. $m^\alpha = m$ for every line m that passes through C.



- Pick a point X not on ℓ and consider the line XX^α
- let $C = \ell \cap XX^\alpha$. The $C^\alpha = C$ since $C \, I \, \ell$
- The line $XX^\alpha = m$ is fixed by α, <u>ie</u> $m^\alpha = m$. Why?
- $m^\alpha = (XC)^\alpha$
  $= X^\alpha C^\alpha$
  $= X^\alpha C$
  $= m$
- Pick any line (other than ℓ or m) that passes through C, call it t.
- <u>Claim</u>: $t^\alpha = t$
- <u>Proof</u>: Suppose $Y \, I \, t$. It's enough to show $Y^\alpha \, I \, t$
- $Y \, I \, t$

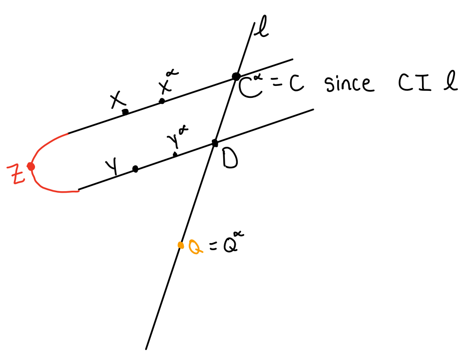
  $Y = XQ \cap t$

  $Y^\alpha = X^\alpha Q \cap t^\alpha$

  $Y^\alpha \, I \, t$?

Setting it up differently:

- We know that ℓ and $m = XX^\alpha$ are fixed by α.
- Now pick a Y not on ℓ or m
- If $YY^\alpha \, I \, C$, ok! ✓



- If $YY^\alpha \not I \, C$, the $YY^\alpha \cap \ell$ at some point $D \neq C$
- XX^α and YY^α have to intersect at some point $Z \not I \, \ell$.
- We are in the case where no point of ℓ is fixed.

- But $Z^{\alpha} = (XC \cap YD)^{\alpha}$
$= X^{\alpha}C \cap Y^{\alpha}D$
$= XC \cap YD$
$= Z$

$\therefore$ contradiction.

- Hence, the assumption that $YY^{\alpha} \not{I} C$ must be wrong.
- Thus, $t^{\alpha} = (YY^{\alpha})^{\alpha}$
$= (YC)^{\alpha}$
$= Y^{\alpha}C$
$= YC$
$= t$
- The other direction (that central => axial) works the same way with the roles of points & lines interchanged.

A (P, ℓ) central collineation α is one that has centre P and axis ℓ.

<u>Fact</u>: If α & β are both (P, ℓ) central collineations, then so is $\alpha\beta = \beta \circ \alpha$

<u>Fact 2</u>: If α is a (P, ℓ) central collineation, so is α^{-1}.