Recall: • A collineation of a projective plane is a function α that takes points to points, lines to lines, is 1-1, onto, and preserves incidence.
$P^\alpha \mathrm{I}\, \ell^\alpha \iff P \,\mathrm{I}\, \ell$

• A collineation of a projective plane is a central collineation if there is a point P s.t if $P \,\mathrm{I}\, \ell$, then $\ell^\alpha = \ell$ [so $P^\alpha = P$]
centre of α

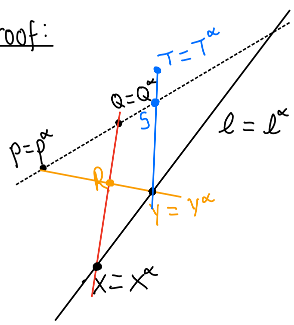
• Axial collineation if there is a line ℓ s.t if $P \,\mathrm{I}\, \ell$, then $P^\alpha = P$ [so $\ell^\alpha = \ell$]
↳ axis of ℓ.

Aim: Show that α is centrall iff α is axial.

Note: Not every collineation will axial or central
eg the collineation of the real projective plane given by $(a,b,c)^\alpha = (b,c,a)$ and $[m,n,k]^\alpha = [n,k,m]$
fixes $(1,1,1)$ and $[1,1,1]$ but no other point or line.

Lemma: Suppose α is a collineation that has axis ℓ and there are two points P and Q with $P \,\not\mathrm{I}\, \ell$ & $Q \,\not\mathrm{I}\, \ell$, (and $P \neq Q$) such that $P^\alpha = P$ and $Q^\alpha = Q$. Then $\alpha = i$ is the identity collineation.

Proof:



• Pick a point R that is not on ℓ and not on PQ
• Claim: $R^\alpha = R$
• Let PR and QR intersect ℓ in the points Y and X, respectively.
• Then

$$\begin{aligned} R^\alpha &= (PY \cap QX)^\alpha \\ &= (PY)^\alpha \cap (QX)^\alpha \\ &= P^\alpha Y^\alpha \cap Q^\alpha X^\alpha \\ &= PY \cap QX \\ &= R \end{aligned}$$

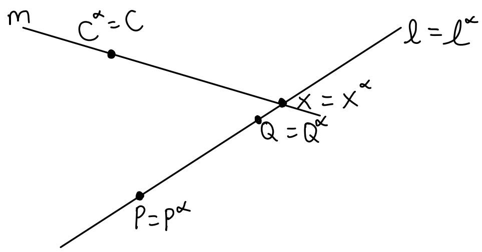
• Pick a point S on PQ such that $S \neq P$, $S \neq Q$, and $S \neq \ell \cap PQ$
• Connect S to a point Y on ℓ (other than $P \cap PQ$)
• let T be a point on SY other than S on Y (so T is not on ℓ and not PQ).
• Thus, $T^\alpha = T$ by our previous work.
• Thus, $$\begin{aligned} S^\alpha &= (PQ \cap TY)^\alpha \\ &= (PQ)^\alpha \cap (TY)^\alpha \\ &= P^\alpha Q^\alpha \cap T^\alpha Y^\alpha \\ &= PQ \cap TY \\ &= S \end{aligned}$$

$\therefore$ $Z^\alpha = Z$ for every point Z, and it follows that $m^\alpha = m$ for every line m too.

Theorem: A central collineation is also axial, and vice versa.
ie If a collineation has an axis it must have a centre, and vice versa.

Proof: Let's assume that a collineation α has axis ℓ. ie $P I \ell \Rightarrow P^{\alpha} I P$

To Show: α has a centre



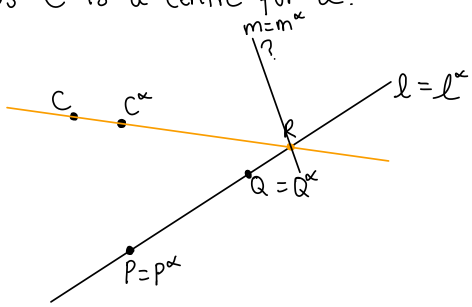
Case 1: There is some point C not on ℓ such that $C^{\alpha} = C$.

Claim: Every line through C is fixed by α.

- Suppose m is a line s.t $C I m$
- Let $X = m \cap \ell$, then $X I \ell$ so $X^{\alpha} = X$

$$m^{\alpha} = (CX)^{\alpha} = C^{\alpha} X^{\alpha}$$
$$= CX$$
$$= m$$

Thus C is a centre for α.



Case 2: there is no point C not on ℓ such that $C^{\alpha} = C$.
ie $C \not{I} \ell$
$\Rightarrow C^{\alpha} \neq C$

Note that in this case any centre of α is on the axis ℓ.

- Let C be any point not on ℓ.
- Let CC^{α} intersect ℓ at R.
- Claim: R is the centre of α. ie For every m with $R I m$, we have $m^{\alpha} = m$.

Come back Monday!